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TWO-CHANNEL LINEAR-PREDICTIVE SPECTRAL ANALYSIS PROGRAM FOR THE--ETC(U)
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Two-Channel Linear-Predictive Spectral Analysis; Program for the HP 9845 Desk Calculator

Albert H. Nuttall
Surface Ship Sonar Department

Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing"; Principal Investigator, Dr. A. H. Nuttall, Code 3302; Program Manager, CAPT D. F. Parrish, Naval Material Command, MAT 08L.

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William A. Von Winkle

W. A. Von Winkle
Associate Technical Director
for Technology

The author of this report is located at the New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320.

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Autoregressive Process	Short Data Segments
Basic Program	Spectral Analysis
Coherence Estimation	Tonal Interference
Cross Spectra	

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A program for linear-predictive spectral analysis of two channels of data, including estimation of the auto-spectra, cross-spectrum, magnitude-squared-coherence, and the argument of the coherence, is presented in Basic for the HP 9845 desk calculator. Timing results for the major subroutines are included, and their dependence on the fundamental parameters of the data, filter, and desired spectral resolution is indicated. This technique and program for spectral analysis are very appropriate for short data segments. In particular, a positive-definite spectral matrix estimate is guaranteed.

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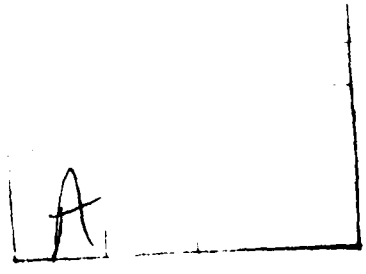
20. Abstract (Cont'd)

Applications to examples including a strong tonal interference in one channel are made, and a possible shortcoming of the technique is pointed out. A suggested remedy is proposed and a philosophy for multichannel spectral analysis is suggested for further consideration.



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List of Symbols

N	Number of data points in each process
Pmax	Maximum order of predictive filter to be considered
Nfft	Size of FFT to be used in spectral analysis
AIC	Akaike Information Criterion
Pbest	Best order of predictive filter to use, based on AIC
Δ	Time sampling increment
f	Frequency
f_N	Nyquist frequency, $(2\Delta)^{-1}$
$x_1(k)$	k-th sample of process 1
$x_2(k)$	k-th sample of process 2
$w_n(k)$	Additive white noise (n = 1, 2)
P	Predictive filter order
$A_P^{(P)}$	Partial forward correlation coefficient of order P
R_o	Covariance matrix at zero delay

Two-Channel Linear-Predictive Spectral Analysis; Program for the HP 9845 Desk Calculator

Introduction

Spectral analysis of short data segments by the standard FFT procedure is not a viable approach; unstable and/or coarse estimates of the spectra result. An attractive technique in this case is linear-predictive spectral analysis, both for the single-channel as well as the multiple-channel cases. See references 1-9, particularly references 7-9 which derive and give Fortran programs for a multiple-channel linear-predictive spectral analysis technique that is a generalization of Burg's technique for the single-channel case (reference 1).

The purpose of this report is twofold: first, we translate the Fortran program in reference 9 into Basic for use on the Hewlett-Packard HP 9845 Desk Calculator, and in the process, also make some minor improvements and modifications to the format and printout statements. We also limit consideration to the two-channel case and thereby take advantage of some simplifications in computing possible for this special case. Second, we apply the program to a pair of stationary processes, one of which has pure tones that are not present in the other process. In this manner, we point out a possibly deleterious effect on the auto-spectral estimates and the coherence estimate, and indicate a method for circumventing some of the difficulty. As a byproduct, a philosophy for multichannel spectral analysis is suggested for further consideration.

Use of Program For Spectral Analysis

In appendix A, the listing for the two-channel linear-predictive spectral analysis technique is presented. Inputs required of the user are the following:

N	Number of data points in each process	
Pmax	Maximum order of predictive filter to be considered	
Nfft	Size of FFT to be used in spectral computation.	(1)

In addition to these integer inputs, the user must modify the subroutine SUB Data (N,X(*)) to accommodate and read in his particular two-channel data. All data are presumed real.

The program computes the (sample) means of each of the two processes and subtracts the means from the data. (Some possible ramifications of this procedure are considered in reference 6, appendix B; in addition, the effect of choosing too small an FFT size, Nfft, is discussed in reference 6.) Next, the covariance matrix (at zero delay) of the input data is computed, and the Akaike Information Criterion (AIC, reference 8, pages 42-44) is evaluated and used to select the integer

Pbest	Best order of predictive filter to use.	(2)

The forward and backward partial correlation coefficients (references 7-9) are evaluated through order Pmax, as well as the forward predictive filter coefficients for Pbest. The normalized correlation matrices are computed through Pmax (extrapolated values beyond Pbest) and the spectral density matrix is computed (via an FFT) from zero to Nyquist frequency, $f_N = (2\Delta)^{-1}$, where Δ is the time-sampling increment of the processes. A partial check on the adequacy of the FFT size, Nfft, is afforded by a printout of the areas under the spectral estimates and comparison with the (sample) covariances of the input data. Finally, the inverse FFT of the spectral estimate gives the aliased normalized correlation matrices; the motivation and equations for this approach are given in reference 9.

A sample printout for a short data sequence (20 data points in each process) is given after the program listing in appendix A, as a test or check case on a user-written program. Also, plots of the corresponding auto-spectral estimates and the coherence estimates are given there for completeness, although this example has no real physical significance.

Timing Results

Execution times for the five major subroutines,

Pcc	Partial correlation coefficients	
Pfc	Predictive filter coefficients	
Peftf	Predictive error filter transfer function	
Sdm	Spectral density matrix	
Acm	Aliased correlation matrices,	(3)

are given in tables 1-5 below, for the HP 9845B Desk Calculator equipped with the Fast Processor Upgrade Kit. Only those variables utilized in each subroutine are considered in these tables, since execution time is independent of the other variables; for example, the execution time of subroutine Pcc does not depend on Pbest.

Table 1. Execution Times for Subroutine Pcc

N	Pmax	Seconds
20	6	1.9
50	10	5.4
100	5	4.7
100	10	9.5
100	15	14.1
1000	47	404.2

Table 2. Execution Times for Subroutine Pfc

Pmax	Pbest	Seconds
5	1	.09
10	1	.15
6	4	.24
15	5	.62
15	11	1.41
47	12	4.06

Table 3. Execution Times for Subroutine Peftf

Pbest	Nfft	Seconds
4	256	17.5
11	256	17.8
1	512	32.0
5	512	32.8
1	1024	63.9
11	1024	66.6

Table 4. Execution Times for Subroutine Sdm

Nfft	Seconds
256	8.9
512	17.7
1024	35.3

Table 5. Execution Times for Subroutine Acm

Nfft	Seconds
256	9.8
512	18.6
1024	37.7

From these tables, we are able to extract the following fairly accurate rules of thumb: the execution time of

Pcc is linearly dependent on N and Pmax
Pfc is linearly dependent on Pmax and Pbest
Peftf is linearly dependent on Nfft, but is essentially independent of Pbest
Sdm is linearly dependent on Nfft
Acm is linearly dependent on Nfft. (4)

These rules allow extrapolation to other cases of interest to the user. The execution times of the FFT itself are given in table 6.

Table 6. Execution Times for Subroutine Fft10

Nfft	Seconds
128	2.6
256	4.5
512	8.4
1024	17.1

If the user is interested only in obtaining the predictive filter coefficients (for example, to do time domain prediction and signal processing), these results are available immediately after execution of subroutine Pfc. There is then no need to resort to the frequency domain routines that follow Pfc; in this manner, execution time and storage can be significantly reduced. An additional reduction in execution time is available by declaring all the loop counters in a subroutine to be INTEGER.

Application to Processes With Tones

Our first example is the two-channel case given numerically by the sample values in reference 7, page 17; reference 8, page K-12; and in reference 9, page D-18. The analytic expression for the autoregression is

$$\begin{aligned} x_1(k) &= .85\, x_1(k-1) - .75\, x_2(k-1) + w_1(k) \\ x_2(k) &= .65\, x_1(k-1) + .55\, x_2(k-1) + w_2(k) \end{aligned} \tag{5}$$

where $\{w_1(k)\}$ and $\{w_2(k)\}$ are uniformly distributed, independent white noise processes with zero means and variances 1/12. General filter and spectral relations for moving-average and autoregressive processes are given in appendix B; these general relations are then specialized to this particular numerical example. It is shown in (B-31) et seq. that the auto spectrum of process $\{x_1(k)\}$ has *four poles and three zeros* in the finite z-plane, even though the two-channel recursion, (5), is only first-order regressive.

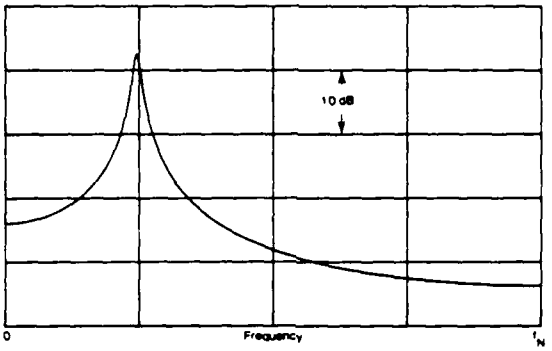
Generally, for a two-channel P-th order regression and independent white excitations (i.e., $E_k = 0$ for $k > P$, $F_k = I\,\delta_{ko}$, and $Q(z) = \Delta I$ in (B-18)), the auto- and cross-spectra of the processes each possess 4P poles and 3P zeros in the finite z-plane (of which P zeros occur at the origin). This is in contrast to the single-channel case, where 2P poles (and a P-th order zero only at the origin) can occur. This increased generality can be anticipated by the observation that whereas a single-channel approximation requires estimation of only P parameters, an M-channel approximation requires estimation of M^2P parameters (4P for the two-channel case $M = 2$). Of course, for a fixed number, N, of data points from each process, the estimation of an increased number of parameters can only be done with increased variance; this is a manifestation of the tradeoff between resolution and stability that accompanies all spectral analysis techniques.

The first-order forward partial correlation coefficient for two-channel process (5) is

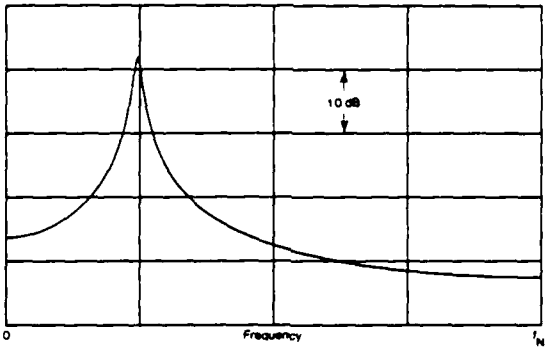
$$A_1^{(1)}(\text{true}) = \begin{bmatrix} .85 & -.75 \\ .65 & .55 \end{bmatrix} , \tag{6}$$

and all other higher-order coefficients are zero. The exact auto spectrum of the first process, $\{x_1(k)\}$, is shown (in dB) in figure 1A; the auto spectrum of the second process, $\{x_2(k)\}$, is shown in figure 1B; the magnitude-squared coherence is displayed in figure 1C; and the argument of the complex coherence or cross spectrum is depicted in figure 1D. There is seen to be a strong narrowband component at approximately one-fourth of the Nyquist frequency $f_N = (2\Delta)^{-1}$, where Δ is the time-sampling increment for the two-channel process (5). This leads to a peak magnitude-squared coherence value of .999013 at $2f\Delta = .2459$.

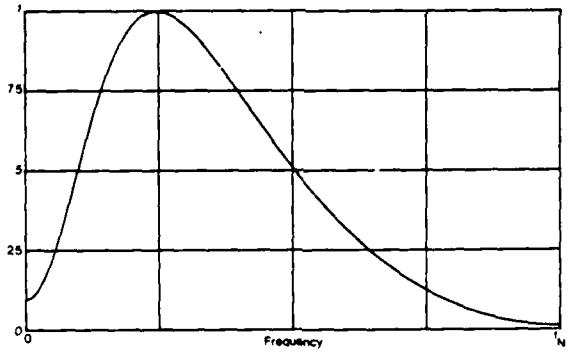
The results of applying the two-channel spectral analysis program in appendix A to the numerical data cited above, with $N = 100$, are shown in figure 2, where the



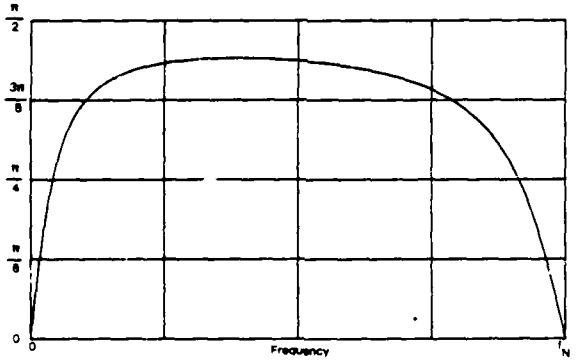
1A. Auto Spectrum of First Process



1B. Auto Spectrum of Second Process

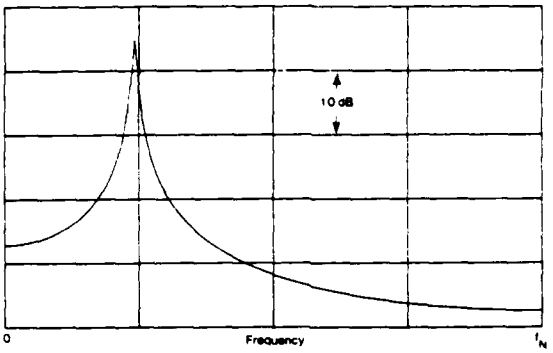


1C. Magnitude-Squared Coherence

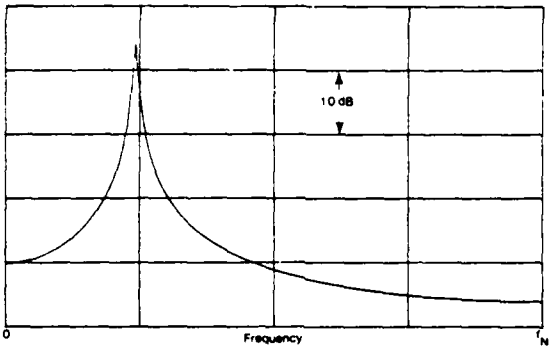


1D. Argument of Complex Coherence

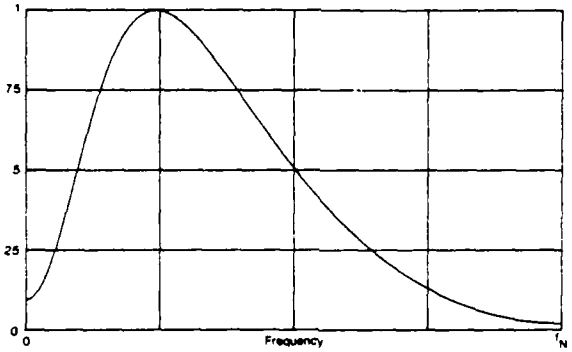
Figure 1. True Spectral Characteristics With No Tone



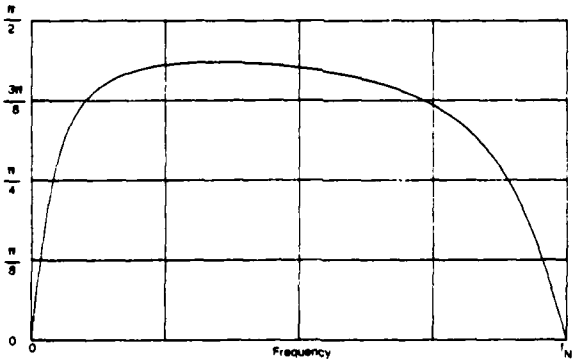
2A. Auto Spectrum of First Process



2B. Auto Spectrum of Second Process



2C. Magnitude-Squared Coherence



2D. Argument of Complex Coherence

Figure 2. Spectral Estimates for N = 100, Pmax = 15, Pbest = 1, No Tone

four parts of this figure correspond directly to those of figure 1. Pbest turns out to be equal to the correct value 1, and the spectral estimates are all quite good. In fact, the estimated magnitude-squared coherence reaches a peak value of .999745 versus the true value of .999013.

The covariance matrix of the process generated by (5) is (reference 8, page 18, after scaling by variance 1/12)

$$R_o(\text{true}) = \begin{bmatrix} 2.095 & 0.405 \\ 0.405 & 1.804 \end{bmatrix} . \tag{7}$$

The corresponding matrix estimate yielded by the program here, based on the particular N = 100 data values cited above, is

$$R_o = \begin{bmatrix} 4.62 & .916 \\ .916 & 3.80 \end{bmatrix} \text{ for N = 100;} \tag{8}$$

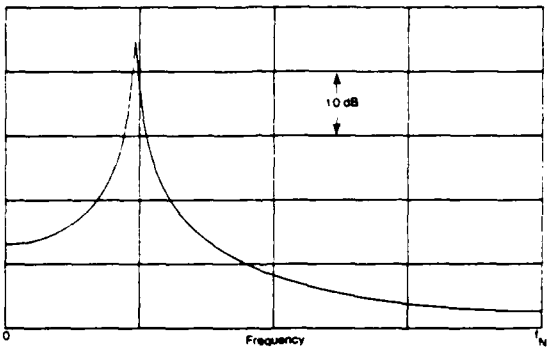
these values are approximately 2.2 times larger than (7), due to the fact that (5) is a narrowband process and the particular 100 pairs of samples used in the spectral estimates happen to lie on a local peak of the instantaneous waveforms. Although the local estimates of the absolute power levels are off considerably, the estimate of the forward partial correlation coefficient is very good; we find, instead of (6),

$$A_1^{(1)} = \begin{bmatrix} .872 & -.770 \\ .634 & .560 \end{bmatrix} \text{ for N = 100.} \tag{9}$$

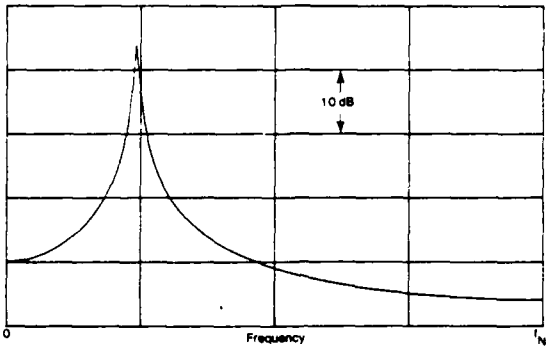
Next we add a pure tone only to the second process $\{x_2(k)\}$ at a frequency equal to 0.6 of the Nyquist frequency, i.e., at $0.6f_N$. The power in the tone is 1/512, i.e., 32.9 dB below the average power, 3.80, in this particular segment of autoregressive process $\{x_2(k)\}$; see (8). The resultant spectral estimates are shown in figure 3; they are virtually identical to figure 2. The only inadequacy of figure 3 is that the autospectral estimate in figure 3B gives no indication of the added tone; of course, there should ideally be no indication of the tone in figure 3A for the auto spectrum of $\{x_1(k)\}$. The value of Pbest was again 1, as determined by the AIC.

When the tonal power in the second process is increased to -26.9 dB, Pbest increases to 4 (see figure 4) and there are humps in both auto-spectral estimates near the tone frequency $0.6f_N$. The coherence estimates (magnitude and argument) are significantly perturbed in a considerable neighborhood of $0.6f_N$; this broad frequency-perturbation width is due to a small value of Pbest having been selected by the AIC.

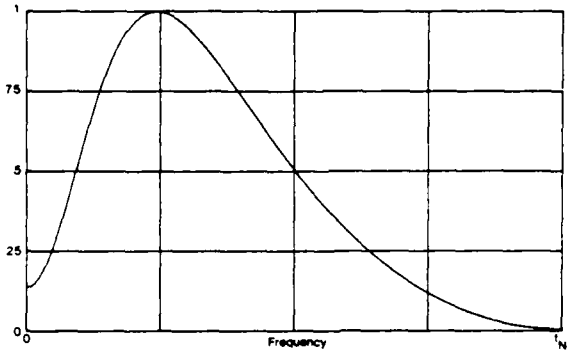
Increasing the tonal power to -20.8 dB results in the estimates depicted in figure 5. Now there is a considerable indication of the tonal power in figure 5B; however, there is also an *undesirable* indication in figure 5A at frequency $0.6f_N$ in the auto-spectral estimate for process $\{x_1(k)\}$. This "feed-across" is due to the fact that we are working with only N = 100 data samples of each process; with this small a data set, the "best" two-channel linear-prediction is misled into an erroneous indication. It is important to observe at this point that *any* auto-spectral estimate based on



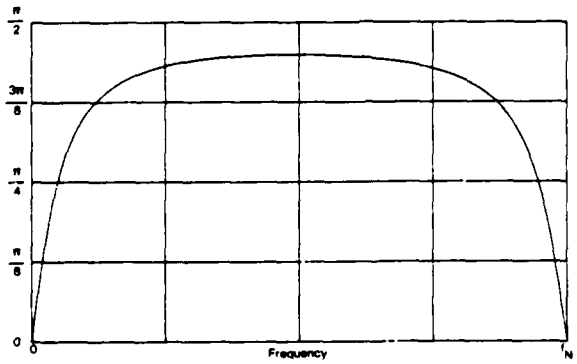
3A. Auto Spectrum of First Process



3B. Auto Spectrum of Second Process

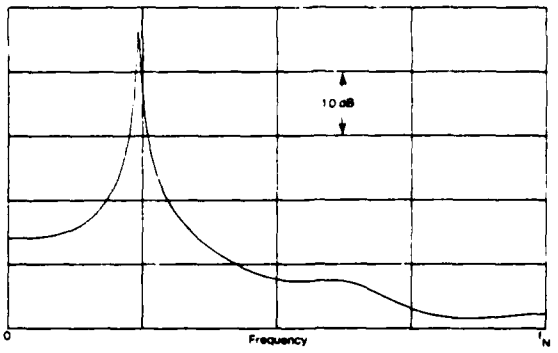


3C. Magnitude-Squared Coherence

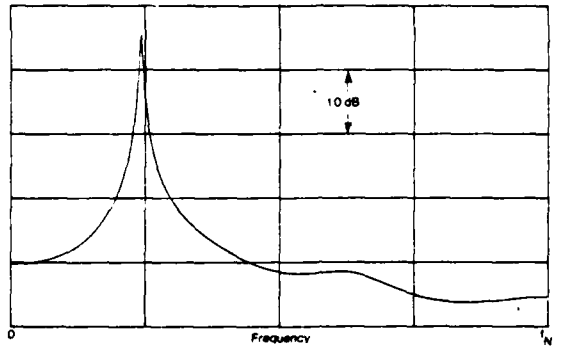


3D. Argument of Complex Coherence

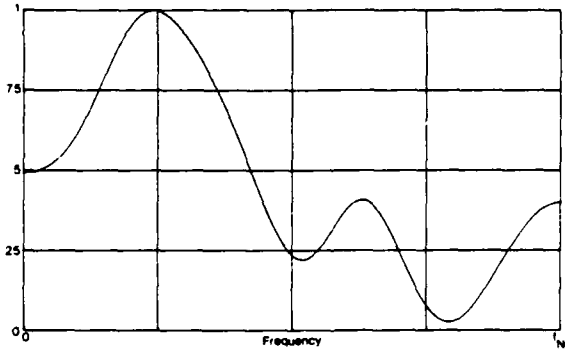
Figure 3. Spectral Estimates for N = 100, Pmax = 15, Pbest = 1, Tone Power = -32.9 dB



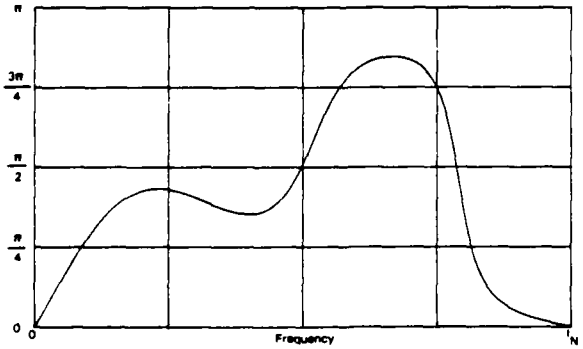
4A. Auto Spectrum of First Process



4B. Auto Spectrum of Second Process

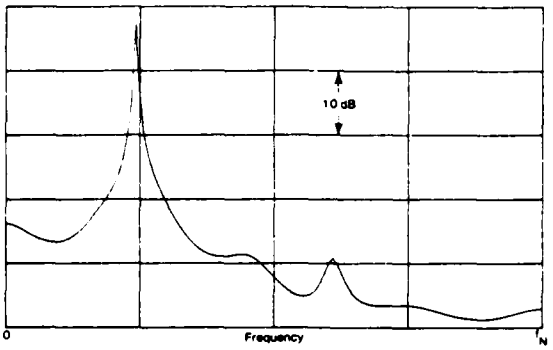


4C. Magnitude-Squared Coherence

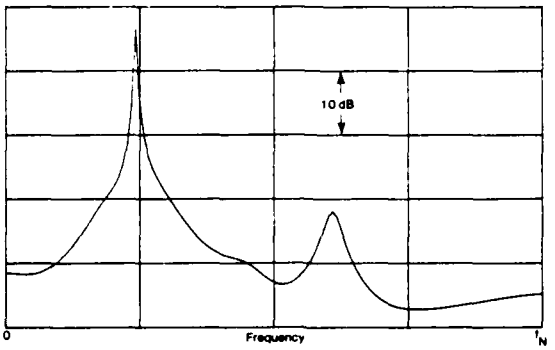


4D. Argument of Complex Coherence

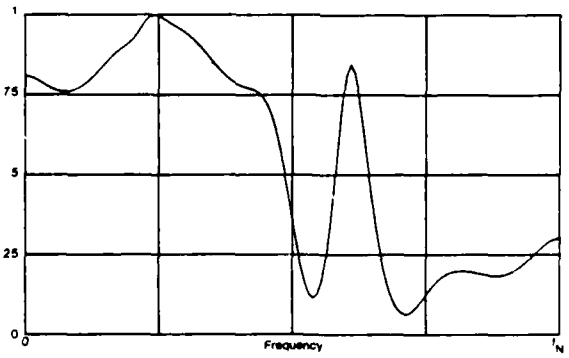
Figure 4. Spectral Estimates for N = 100, Pmax = 15, Pbest = 4, Tone Power = -26.9 dB



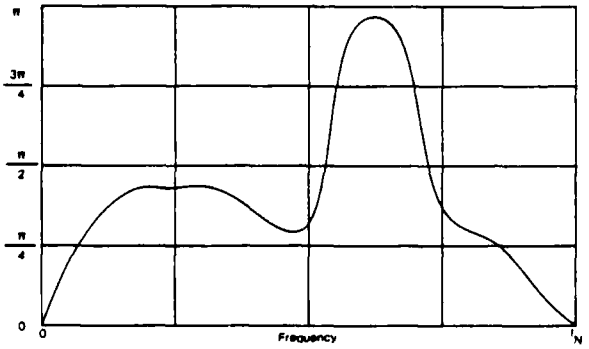
5A. Auto Spectrum of First Process



5B. Auto Spectrum of Second Process



5C. Magnitude-Squared Coherence



5D. Argument of Complex Coherence

Figure 5. Spectral Estimates for N = 100, Pmax = 15, Pbest = 8, Tone Power = -20.8 dB

samples of process $\{x_1(k)\}$ *alone* would not give this tonal indication, since the tone is not present in this process.

The coherence estimates in figure 5 fare no better, even though Pbest = 8 now. A large magnitude-squared coherence value of 0.85 is yielded at frequency $0.6f_N$. The progression towards poorer behavior is also present in figure 6, which employs a tonal power of -14.8 dB relative to the sample power in $\{x_2(k)\}$. Now the undesired peak magnitude-squared coherence estimate is 0.9. A tonal power of -8.8 dB (figure 7) yields a near-unity magnitude-squared coherence estimate at $0.6f_N$, and a very substantial tonal indication in the auto spectrum of $\{x_1(k)\}$, figure 7A.

The situation is markedly improved if more data samples are available. When N is increased to 1000, and data are generated via (5) as before, the sample covariance for the particular data set generated is

$$R_o = \begin{bmatrix} 2.60 & .514 \\ .514 & 2.27 \end{bmatrix} \text{ for } N = 1000, \tag{10}$$

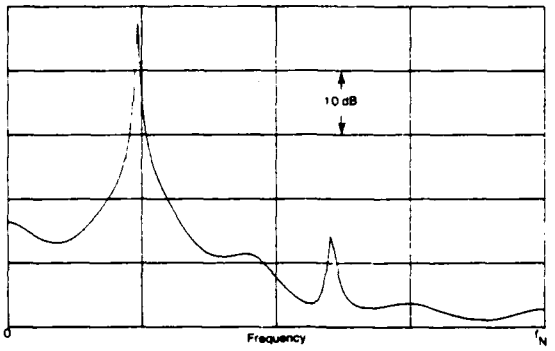
for no tone present. When a tone is added to process $\{x_2(k)\}$, with strength -24.6 dB relative to the sample power, 2.27, of the second process, the resultant spectral estimates are as displayed in figure 8. There is a slight hump at $0.6f_N$ in figure 8B, and a near-zero coherence estimate at this frequency. Recall that the ideal characteristics would be identical to figure 1 except for an impulse in figure 8B at $0.6f_N$ and a very sharp null in the magnitude-squared coherence at $0.6f_N$.

The results in figure 8 were achieved by taking Pmax = 8, for which the AIC indicated Pbest = 8 for this particular data set. However, the AIC is a very flat function of filter order P in this range, and it is difficult to justify a particular value of P as "best". Some additional information about the autoregressive portion of the observed process, such as a limit on P, could be useful; for example, when we specified Pmax as 1, the results were very similar to figure 1. There was virtually no indication of the tone in any of the spectral estimates, even though it was in the $\{x_2(k)\}$ data at a relative level of -24.6 dB with respect to the sample power, 2.27, of the second autoregressive component. In fact, the estimated first partial correlation coefficient was

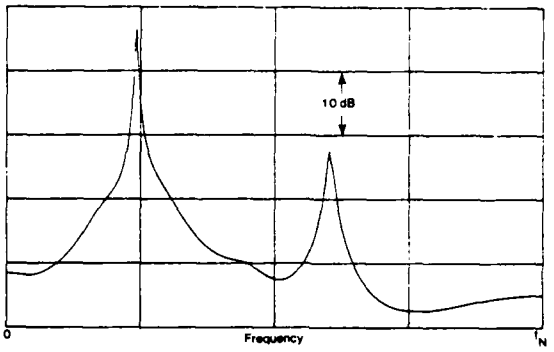
$$A_1^{(1)} = \begin{bmatrix} .8543 & -.7394 \\ .6578 & .5415 \end{bmatrix} \text{ for } N = 1000, \tag{11}$$

which is very close to the true value, (6).

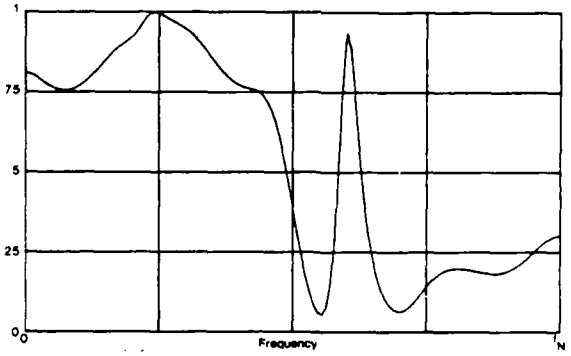
Results for the spectral estimates when the tonal power is increased to -18.6 dB, -12.6 dB, -6.6 dB, and -0.6 dB are given in figures 9, 10, 11, and 12, respectively, all corresponding to Pmax = 8 and Pbest = 8. Even for the nearly 0 dB case in figure 12, there is virtually no indication in auto-spectral estimate 12A of the strong tonal in process $\{x_2(k)\}$, figure 12B. The magnitude-squared coherence estimate in figure 12C appears to have developed a couple of zeros and poles near the frequency $f = 0.6f_N$, where the strong tone is located; recall that we have 4P = 32 poles available in the approximation for Pbest = 8. Typically, it has been observed that a strong tonal present in only one process manifests itself in the



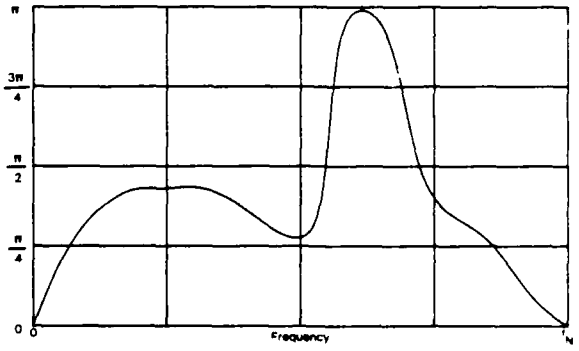
6A. Auto Spectrum of First Process



6B. Auto Spectrum of Second Process

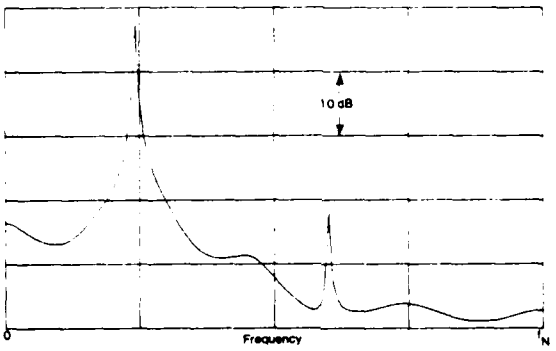


6C. Magnitude-Squared Coherence

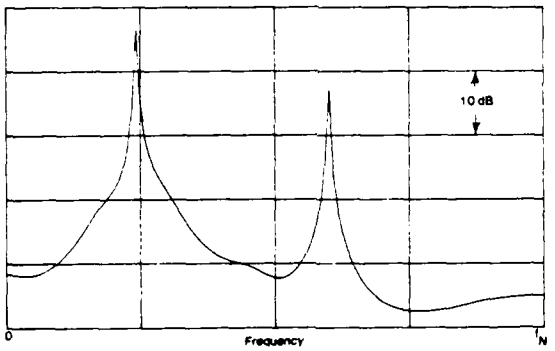


6D. Argument of Complex Coherence

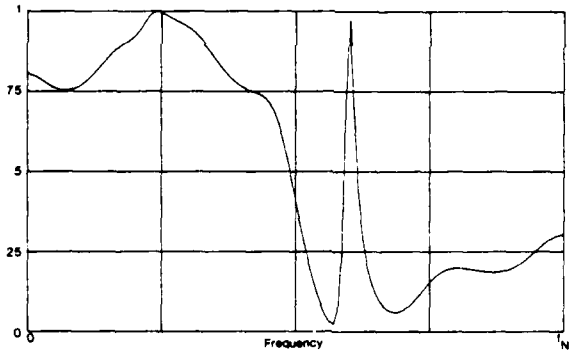
Figure 6. Spectral Estimates for N = 100, Pmax = 15, Pbest = 8, Tone Power = -14.8 dB



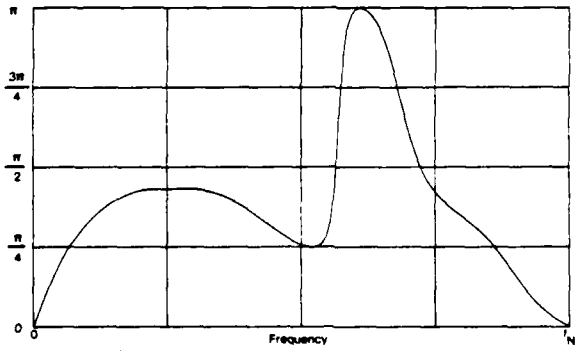
7A. Auto Spectrum of First Process



7B. Auto Spectrum of Second Process

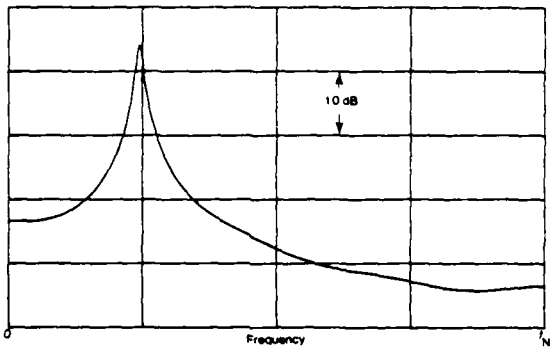


7C. Magnitude-Squared Coherence

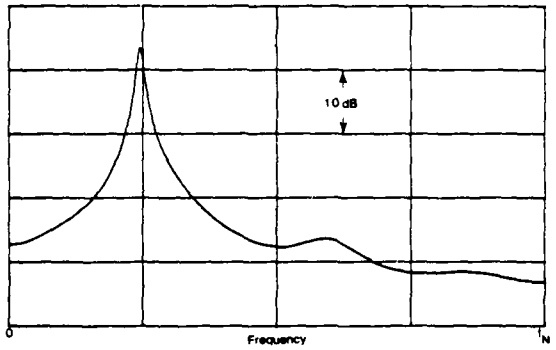


7D. Argument of Complex Coherence

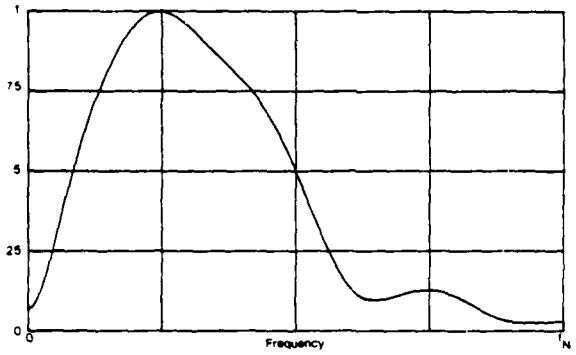
Figure 7. Spectral Estimates for N = 100, Pmax = 15, Pbest = 8, Tone Power = -8.8 dB



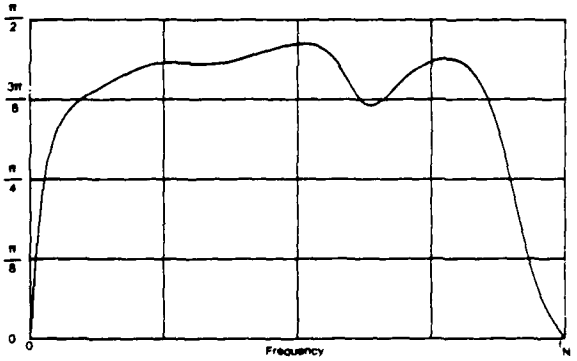
8A. Auto Spectrum of First Process



8B. Auto Spectrum of Second Process

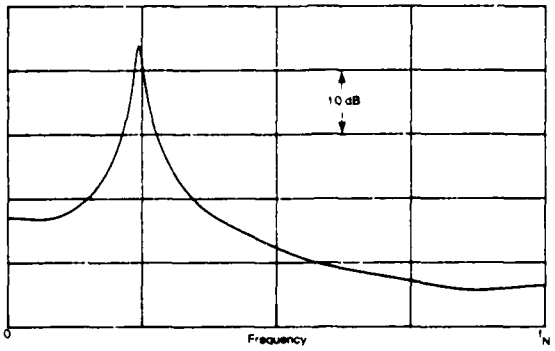


8C. Magnitude-Squared Coherence

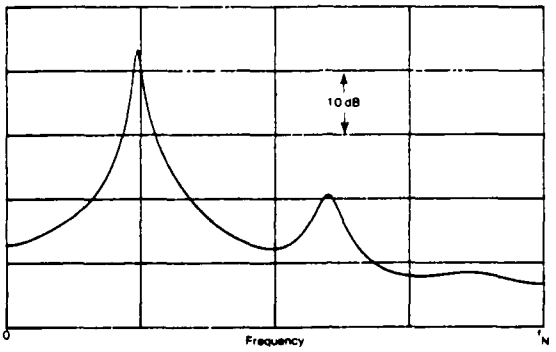


8D. Argument of Complex Coherence

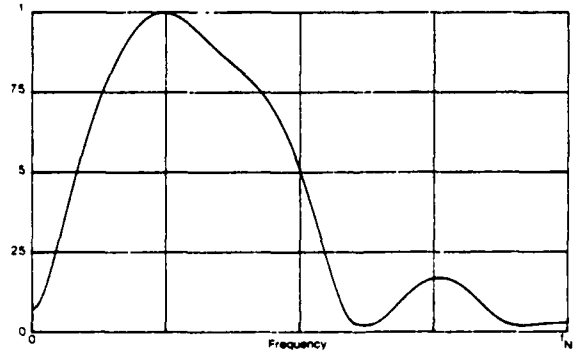
Figure 8. Spectral Estimates for N = 1000, Pmax = 8, Pbest = 8, Tone Power = -24.6 dB



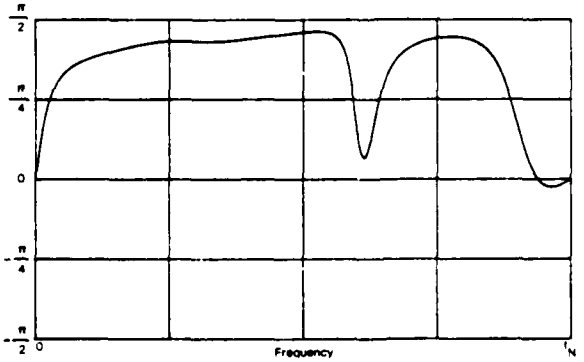
9A. Auto Spectrum of First Process



9B. Auto Spectrum of Second Process

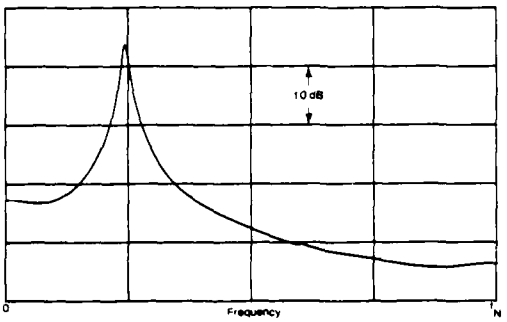


9C. Magnitude-Squared Coherence

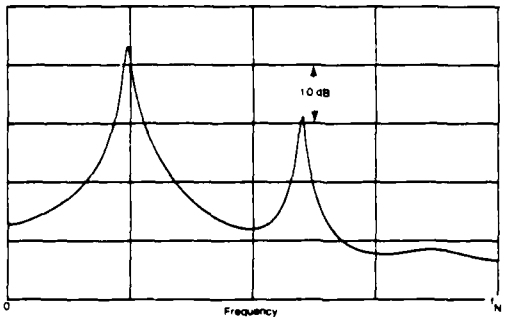


9D. Argument of Complex Coherence

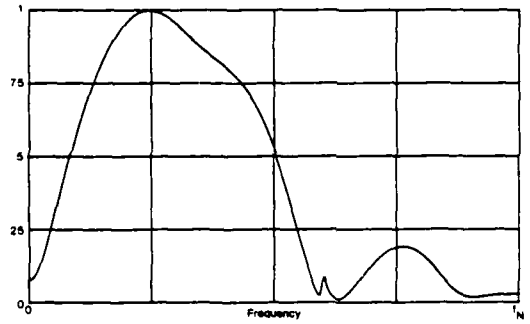
Figure 9. Spectral Estimates for N = 1000, Pmax = 8, Pbest = 8, Tone Power = -18.6 dB



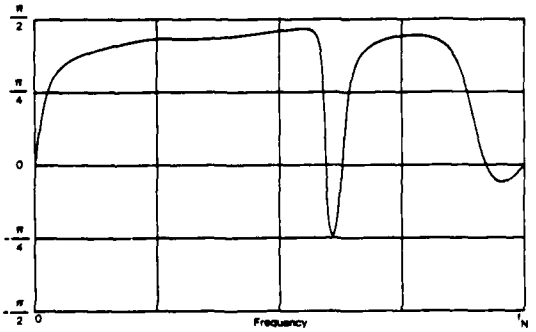
10A. Auto Spectrum of First Process



10B. Auto Spectrum of Second Process

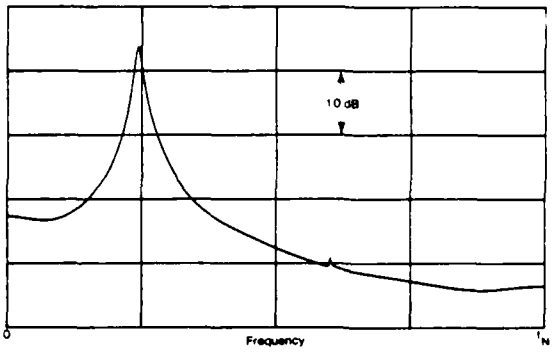


10C. Magnitude-Squared Coherence

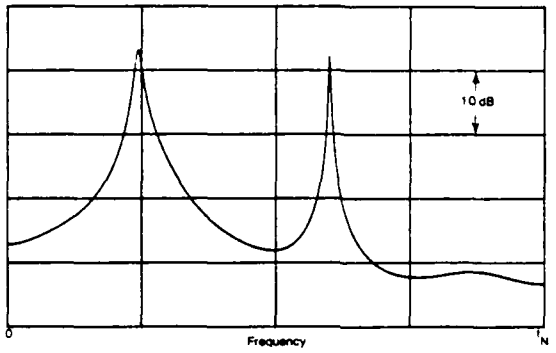


10D. Argument of Complex Coherence

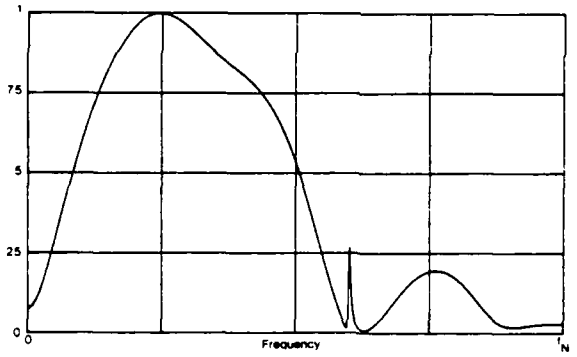
Figure 10. Spectral Estimates for N = 1000, Pmax = 8, Pbest = 8, Tone Power = -12.6 dB



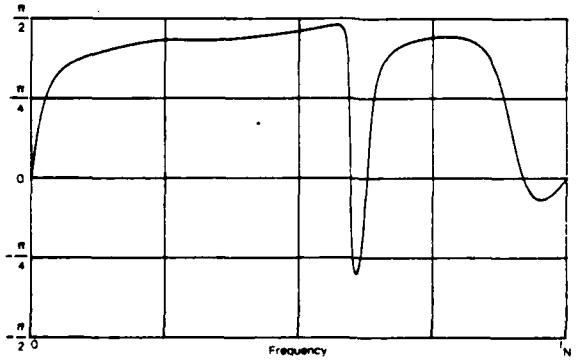
11A. Auto Spectrum of First Process



11B. Auto Spectrum of Second Process

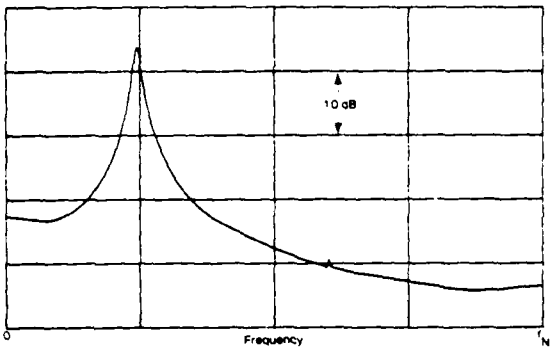


11C. Magnitude-Squared Coherence

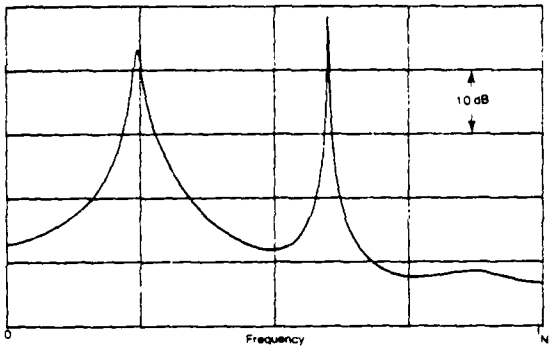


11D. Argument of Complex Coherence

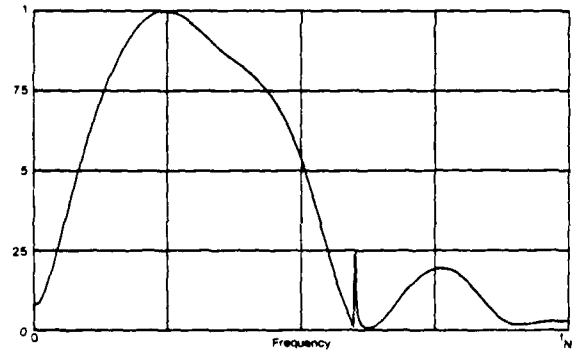
Figure 11. Spectral Estimates for N = 1000, Pmax = 8, Pbest = 8, Tone Power = -6.6 dB



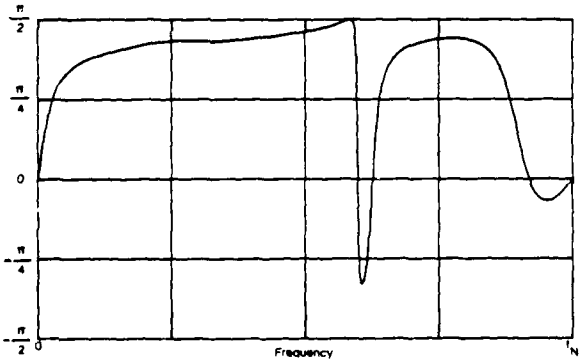
12A. Auto Spectrum of First Process



12B. Auto Spectrum of Second Process



12C. Magnitude-Squared Coherence



12D. Argument of Complex Coherence

Figure 12. Spectral Estimates for N = 1000, Pmax = 8, Pbest = 8, Tone Power = -0.6 dB

coherence estimate as a sharp spike at the tone frequency. The change in argument in figure 12D in the neighborhood of this frequency can serve as an indicator of the number of poles and zeros clustered there.

When Pmax was increased to 47, the AIC yielded Pbest = 25 for these last four figures. However, the spectral estimates for Pbest = 25 proved to be too spiky and erratic. Also the selection of Pbest at 25 is rather tenuous, as figure 13 indicates; this is a plot of the AIC versus filter order P in the range (1, 47). Although the absolute minimum occurs at P = 25, there are significant drops in the curve at P = 4, 6, and 8. Selection of P at one of these significant drops appears to be a promising approach, instead of using the absolute minimum of the curve. In addition, the flatness of the curve is brought out by observing that the range of values of AIC is limited to (-4.80, -4.73) for P in the wide range from 10 to 47. Thus, the local minor drops and rises in the AIC curves are not significant; selection of values of P corresponding to significant decreases seems to be a viable approach.

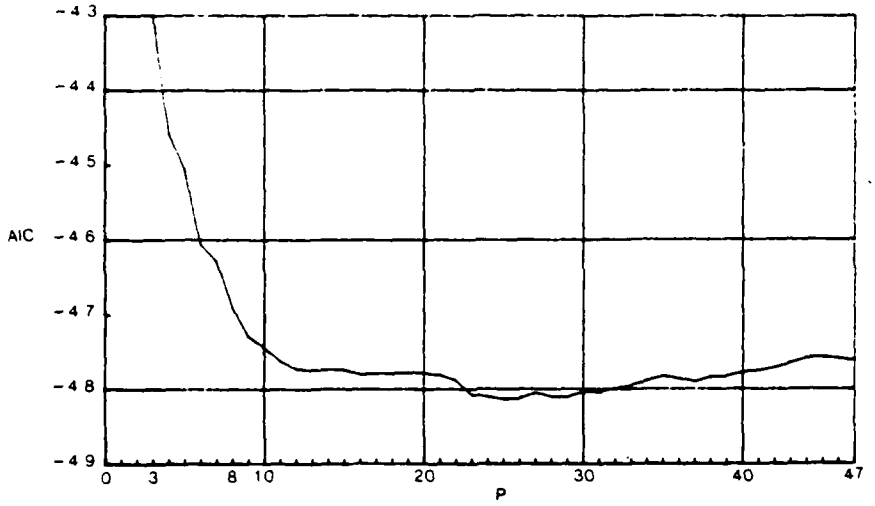
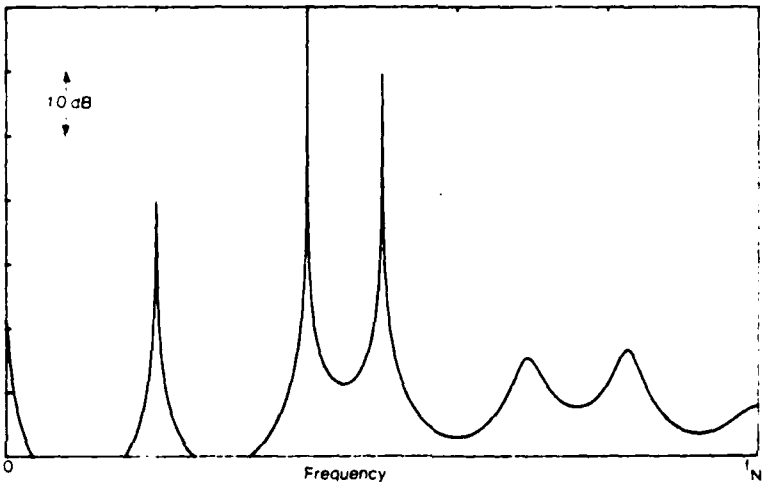


Figure 13. Akaike Information Criterion for N = 1000, Tone Power = -12.6 dB

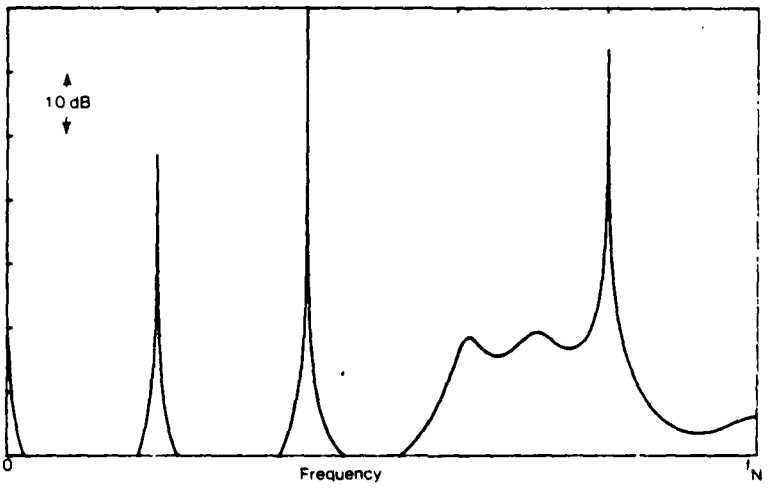
The last example we consider is a two-channel process of N = 64 data points, composed of several tones, some of which are at common frequencies, and some of which are not; this example was supplied by S. L. Marple (reference 10). In particular, process $\{x_1(k)\}$ has two strong tones at $f = 0.4f_N$ and $0.5f_N$, and a weaker tone (-20 dB) at $f = 0.2f_N$, in addition to some low level, colored background noise. The other process has two strong tones at $f = 0.4f_N$ and $0.8f_N$, and a weaker tone (-20 dB) at $f = 0.2f_N$. Thus the tonal frequencies common to both processes are $0.2f_N$ and $0.4f_N$, whereas the uncommon frequencies are $0.5f_N$ and $0.8f_N$. The two auto-spectral estimates of each process (obtained via the single-channel, forward-backward averaging technique of reference 4) are displayed in figure 14 for prediction length P = 12 (24 poles for each spectral estimate). There is, of course, no cross-feed at frequencies $0.5f_N$ and $0.8f_N$.

The spectral estimates of the same two-channel data (obtained via the program in appendix A which includes coherence estimation) are given in figure 15. The value

of P used was 6, which allows for 24 poles in each spectral estimate. The low number of data points, N = 64, now allows some undesired cross-feed in figures 15A and 15B at f = $0.8f_N$ and $0.5f_N$, respectively. This also shows up in the magnitude-squared coherence estimate as two very sharp spikes at these two frequencies, whereas the true coherence is zero at these two frequencies. This limited capability of the multi-channel linear predictive technique can be improved by utilizing larger data sets; N = 64 is too small a data size to accomplish a high quality result for a data set such as this with strong interfering tones.

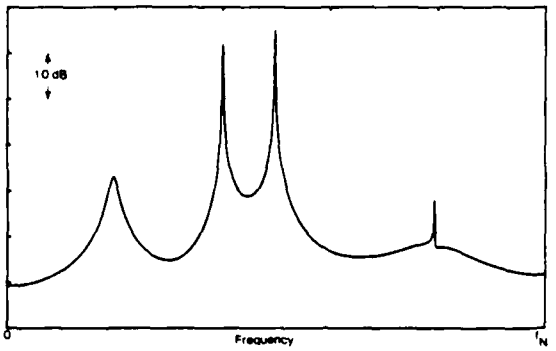


14A. Auto Spectrum of First Process

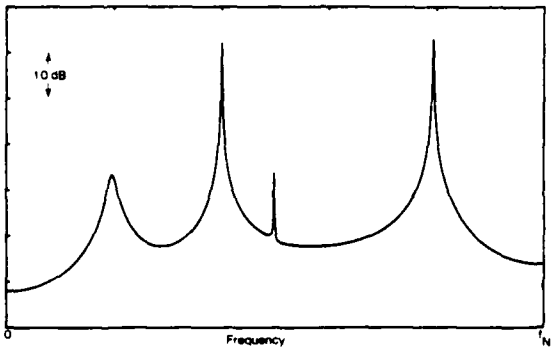


14B. Auto Spectrum of Second Process

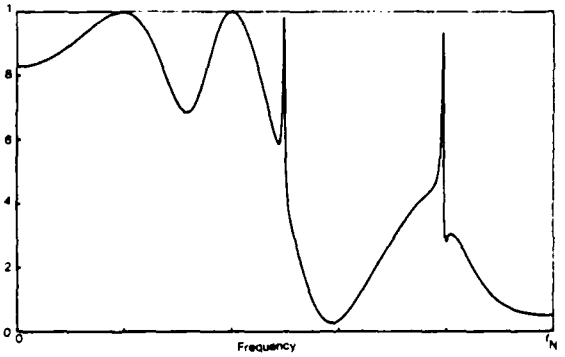
Figure 14. Auto-Spectral Estimates for Multitone Example, N = 64, P = 12



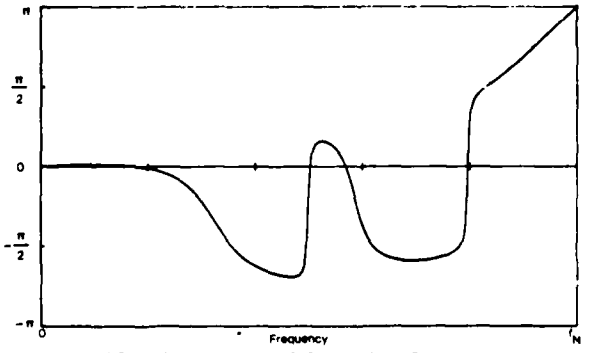
15A. Auto Spectrum of First Process



15B. Auto Spectrum of Second Process



15C. Magnitude-Squared Coherence



15D. Argument of Complex Coherence

Figure 15. Spectral Estimates for Multitone Example, N = 64, P = 6

Discussion and Conclusions

A program for two-channel auto- and cross-spectral estimation has been presented and illustrated for cases including interfering tones. If the number of available points, N, is too small, some misleading estimates may be obtained because of cross-feed between the finite lengths of data from each channel. This cross-feed manifests itself as narrow spurious spikes in the spectral and coherence estimates. Choice of the appropriate value of filter order, P, is possible by observing where the AIC undergoes significant negative jumps, rather than by using the absolute minimum of the curve. It can also be illuminating to overlay plots made with two (or more) different values of P, thereby obtaining different degrees of resolution and stability from the same data set. The recursive nature of the linear-predictive approach makes this practice easy to achieve.

A more fundamental observation about spectral estimation in general is now developed. Suppose we are given finite data records of three stationary processes x(t), y(t), and z(t), and we wish to estimate all the auto spectra and cross spectra involved. The Blackman and Tukey and weighted-FFT approaches evaluate the auto spectrum of each process separately. Thus, the spectrum of x(t) is estimated without interference from y(t) and z(t); the availability of the data records for y(t) and z(t) plays no part in the eventual auto-spectral estimate for x(t). Additionally, the cross-spectral estimate for processes x(t) and y(t) is independent of the available data on the z(t) process. Finally, the coherence estimate between two processes is independent of any additional data records for other (statistically related) processes.

On the other hand, the generalization (in references 7-9) of Burg's single-channel linear-predictive spectral analysis approach to the multichannel case gives auto-spectral estimates of the x(t) process that are dependent on the available values of y(t) and z(t). Also, the cross-spectral estimate between x(t) and y(t) is dependent on the particular z(t) data available. This procedure can be poor for short data lengths if, for example, y(t) contains a strong tone at f_o that is not present in x(t) or z(t). Thus, estimates of spectra $G_{xx}(f)$, $G_{xy}(f)$, and $G_{xz}(f)$ all contain tonal indications at f_o that should not be there. These spurious tonal indications are due to cross-feed between the available finite data segments of the various processes.

This raises the following questions:

- Should the estimate of $G_{xx}(f)$ be determined only from the available x(t) data record ?
- Should the estimate of $G_{xy}(f)$ be determined only from the available x(t) and y(t) data records ?
- If coherence $C_{xy}(f_o) = 0$, why use y(t) to estimate $G_{xx}(f_o)$?
- If coherence $C_{xy}(f_o) = 1$, why use the completely statistically dependent y(t) data to estimate $G_{xx}(f_o)$?

This philosophy of discarding "irrelevant" data would be consistent with the Blackman and Tukey and FFT approaches. Carrying this philosophy on, we are led to the following: estimate $G_{xx}(f)$ solely from the x(t) data by some good single-

channel linear-predictive technique, such as forward-and-backward averaging, coupled with an efficient way of inverting the relevant matrices (e.g., references 4 and 5). Then estimate cross spectrum $G_{xy}(f)$ or coherence $C(f)$ directly, by some (yet unknown) linear predictive technique whose sole goal is linear prediction of $x(t)$ from $y(t)$ and vice versa, with no interest in or diversion from simultaneous estimation of $G_{xx}(f)$ or $G_{yy}(f)$. By this means, we can concentrate on extracting all the relevant cross-spectral information with maximum stability and resolution. Other cross spectra of interest between particular pairs of available processes can be similarly obtained, one at a time. This procedure is currently under investigation.

Appendix A

Program For Two-Channel Linear-Predictive Spectral Analysis

The program listing below in Basic for the HP 9845B Desk Calculator is a translation and update of that given in references 7-9. A complete breakdown and explanation of the components and subroutines of the program are given in reference 8, and in reference 9, appendix D.

Inputs required of the user are the integers listed in lines 20, 30, 40; they are

N	Number of data points in each process;
Pmax	Maximum order of predictive filter to be considered;
Nfft	Size of FFT to be used in spectral analysis.

In addition, the user must modify subroutine SUB Data(N,X(*)) in lines 5430-5490 at the end of the program to read in his own particular two-channel data sets. Pbest can be forced to equal Pmax by setting Fac = 0 in SUB Pcc.

An explanation of the program output is given under equation (1) of the main text of this report. A sample printout for a short (N = 20) data sequence that can be used as a check case on the program is presented following the listing below. Sample plots of the auto-spectral estimates and the coherence estimates conclude the appendix.

```
10  ! TWO-CHANNEL LINEAR PREDICTIVE SPECTRAL ANALYSIS, TR 5501 - 5729
20    N=20              ! NUMBER OF DATA POINTS IN EACH PROCESS
30    Pmax=6            ! MAXIMUM ORDER OF PREDICTIVE FILTER
40    Nfft=256          ! SIZE OF FFT
50    OPTION BASE 1
60    REDIM Y(2,N),Z(2,N),Ap(Pmax,2,2),Bp(Pmax,2,2
70    REDIM Rn(Pmax,2,2),Aic(0:Pmax),X11(Nfft),Y11(Nfft),X12(Nfft)
80    REDIM Y12(Nfft),X21(Nfft),Y21(Nfft),X22(Nfft),Y22(Nfft)
90    DIM Y(2,1000),Z(2,1000),Ap(25,2,2),Bp(25,2,2)
100   DIM Rn(25,2,2),Aic(0:25),X11(1024),Y11(1024),X12(1024)
110   DIM Y12(1024),X21(1024),Y21(1024),X22(1024),Y22(1024)
120   DIM Ave(2),Ubest(2,2),U(2,2),V(2,2),U1(2,2),V1(2,2),A(2,2)
130   DIM B(2,2),R(2,2),Wa(2,2),Wb(2,2),Wc(2,2),Wd(2,2),We(2,2)
140   PRINT "NUMBER OF DATA POINTS IN EACH PROCESS N =";N
150   PRINT "MAXIMUM ORDER OF PREDICTIVE FILTER Pmax =";Pmax
160   PRINT "SIZE OF FFT Nfft =";Nfft
170   PRINT
180   CALL Data(N,Y(*))
190   PRINT "PROCESS NUMBER 1"
200   FOR I=1 TO N
210   PRINT Y(1,I);
220   NEXT I
230   PRINT LIN(1)
240   PRINT "PROCESS NUMBER 2"
250   FOR I=1 TO N
260   PRINT Y(2,I);
270   NEXT I
280   PRINT LIN(2)
290   CALL Pcc(N,Pmax,Y(*),Z(*),Ave(*),Wa(*,*),Wb(*,*),Wc(*,*),Wd(*,*),We(*),R(*,*),U(*,*),V
(*),Aic(*),Pbest,Ubest(*),U1(*),V1(*),A(*,*),B(*,*),Ap(*,*,*),Bp(*,*,*)
300   PRINT "MEANS OF INPUT DATA (Ave):"
310   PRINT Ave(1)
320   PRINT Ave(2)
330   PRINT LIN(1)
340   PRINT "COVARIANCE MATRIX OF INPUT DATA (R):",R(*,*)
350   PRINT "AKAIKE INFORMATION CRITERION:"
360   PRINT "  P         Aic(P)"
370   IMAGE 3D,4(4X,M. 9DE)
380   FOR P=0 TO Pmax
390   PRINT USING 370;P,Aic(P)
400   NEXT P
410   PRINT LIN(1)
420   PRINT "Pbest =";Pbest
430   PRINT LIN(1)
440   PRINT "Ubest:",Ubest(*)
450   PRINT "FORWARD PARTIAL CORRELATION COEFFICIENTS:"
460   PRINT "  P         Ap(1,1)             Ap(2,1)             Ap(1,2)
 Ap(2,2)"
470   FOR P=1 TO Pmax
480   PRINT USING 370;P,Ap(P,1,1),Ap(P,2,1),Ap(P,1,2),Ap(P,2,2)
490   NEXT P
500   PRINT LIN(1)
```

```
510   PRINT "BACKWARD PARTIAL CORRELATION COEFFICIENTS:"
520   PRINT "  P         Bp(1,1)              Bp(2,1)              Bp(1,2)
 Bp(2,2)"
530   FOR P=1 TO Pmax
540   PRINT USING 370;P,Bp(P,1,1),Bp(P,2,1),Bp(P,1,2),Bp(P,2,2)
550   NEXT P
560   PRINT LIN(1)
570   IF Pbest=0 THEN 890
580   CALL Pfc(Pmax,Pbest,R(*),Ap(*),Bp(*),Wa(*),Wb(*),Wc(*),Wd(*),Pn(*),R11,R22
,R12)
590   PRINT "FORWARD PREDICTIVE FILTER COEFFICIENTS FOR Pbest:"
600   PRINT "  P         Ap(1,1)              Ap(2,1)              Ap(1,2)
 Ap(2,2)"
610   FOR P=1 TO Pbest
620   PRINT USING 370;P,Ap(P,1,1),Ap(P,2,1),Ap(P,1,2),Ap(P,2,2)
630   NEXT P
640   PRINT LIN(1)
650   PRINT "NORMALIZED CORRELATION MATRICES (Rn):"
660   PRINT "DELAY       AUTO11               CROSS21              CROSS12
 AUTO22"
670   PRINT USING 370;0,R(1,1),R(2,1),R(1,2),R(2,2)
680   FOR P=1 TO Pmax
690   PRINT USING 370;P,Rn(P,1,1),Rn(P,2,1),Rn(P,1,2),Rn(P,2,2)
700   NEXT P
710   PRINT LIN(1)
720   CALL Peftf(Pbest,Nfft,Ap(*),X11(*),Y11(*),X12(*),Y12(*),X21(*),Y21(*),X22(
*),Y22(*))
730   CALL Sdm(Nfft,Ubest(*),Wa(*),Wb(*),Wc(*),Wd(*),We(*),X11(*),Y11(*),X12(*),
Y12(*),X21(*),Y21(*),X22(*),Y22(*),S11,S22,S12)
740   PRINT "SPECTRAL DENSITY MATRIX AND COHERENCE, FROM ZERO FREQUENCY (BIN 1):
"
750   PRINT " BIN    AUTO11        AUTO22     RE(CROSS12)   IM(CROSS12)    MAG SQ COH
    ARGUMENT"
760   IMAGE 3D,5(M.6DE,1X),M.6DE
770   FOR I=1 TO 30
780   L=I
790   IF I<16 THEN 840
800   IF I>16 THEN 830
810   PRINT "***"
820   GOTO 850
830   L=I+Nfft/2-29
840   PRINT USING 760;L,X11(L),X22(L),X12(L),Y12(L),Y11(L),Y22(L)
850   NEXT I
860   PRINT LIN(1)
870   PRINT "TRAPEZOIDAL SUMS OF SPECTRA:"
880   PRINT S11,S22,S12
890   PRINT
900   PRINT "COVARIANCES OF INPUT DATA:"
910   PRINT R11,R22,R12
920   PRINT LIN(1)
930   CALL Acm(Nfft,X11(*),X12(*),Y12(*),X21(*),Y21(*),Y22(*),X11m1,X22m1,X11m0,
X22m0)
940   N1=Nfft+1
950   N2=Nfft/2
960   N22=N2+2
```

```
950   PRINT "ALIASED NORMALIZED CORRELATION MATRICES:"
960   PRINT "DELAY       AUTO11              CROSS21             CROSS12
 AUTO22"
990   PRINT USING 370;0,X11(N22),X21(1),X21(1),X22(N22)
1000  FOR I=1 TO 27
1010  L=I
1020  IF I<16 THEN 1070
1030  IF I>16 THEN 1060
1040  PRINT "***"
1050  GOTO 1080
1060  L=I+N2-29
1070  PRINT USING 370;L,X11(N22+L),X21(N1-L),X21(1+L),X22(N22+L)
1080  NEXT I
1090  PRINT USING 370;N2-1,X11m1,X21(N22),X21(N2),X22m1
1100  PRINT USING 370;N2,X11m0,X21(N2+1),X21(N2+1),X22m0
1110  PRINT LIN(2)
1120  PRINTER IS 0
1130  PRINT "AUTO SPECTRAL DENSITIES IN DB:"
1140  PLOTTER IS "GRAPHICS"
1150  GRAPHICS
1160  SCALE 0,N2,-5,0
1170  GRID N2/4,1
1180  PENUP
1190  FOR I=0 TO N2
1200  PLOT I,LGT(X11(I+1))
1210  NEXT I
1220  PENUP
1230  FOR I=0 TO N2
1240  PLOT I,LGT(X22(I+1))
1250  NEXT I
1260  PENUP
1270  DUMP GRAPHICS
1280  PRINT LIN(3)
1290  PRINT "MAGNITUDE SQUARED COHERENCE AND ARGUMENT"
1300  PLOTTER IS "GRAPHICS"
1310  SCALE 0,N2,0,1
1320  GRID N2/4,.25
1330  PENUP
1340  FOR I=0 TO N2
1350  PLOT I,Y11(I+1)
1360  NEXT I
1370  PENUP
1380  SCALE 0,N2,-PI,PI
1390  FOR I=0 TO N2
1400  PLOT I,Y22(I+1)
1410  NEXT I
1420  PENUP
1430  DUMP GRAPHICS
1440  PRINT LIN(4)
1450  PRINTER IS 16
1460  END
1470  !
```

```
1480  SUB Pcc(N,Pmax,Y(*),Z(*),Ave(*),Wa(*),Wb(*),Wc(*),Wd(*),We(*),R(*),U(*),V(
*),Ai(*),Pbest,Ubest(*),Ui(*),Vi(*),A(*),B(*),Ap(*),Bp(*)
1490  Ia=INT(1.5*SQR(N))
1500  IF Pmax<=Ia THEN 1520
1510  PRINT "Pmax =";Pmax;"IS TOO LARGE FOR N =";N;" SEARCH LIMITED TO P =";Ia
1520  Ia=MIN(Ia,Pmax)
1530  Fac=8/N     ! Fac=0  WOULD FORCE Pbest EQUAL TO Pma
1540  MAT Ave=RSUM(Y)
1550  MAT Ave=Ave/(N)
1560  A1=Ave(1)
1570  A2=Ave(2)
1580  FOR I=1 TO N
1590  Y(1,I)=Y(1,I)-A1
1600  Y(2,I)=Y(2,I)-A2
1610  NEXT I
1620  MAT Z=Y
1630  CALL Auto(2,N-1,Y(*),Wc(*))
1640  CALL Auto(1,1,Y(*),Wd(*))
1650  CALL Auto(N,N,Y(*),We(*))
1660  MAT Wa=Wc+We
1670  MAT Wb=Wc+Wd
1680  MAT R=Wb+We
1690  MAT R=R/(N)
1700  MAT U=R
1710  MAT V=R
1720  CALL Cross(2,N,Y(*),Z(*),Wc(*))
1730  Aic(0)=LOG(DET(U))
1740  Aicmin=Aic(0)
1750  Pbest=0
1760  MAT Ubest=U
1770  FOR P=1 TO Ia
1780  MAT Vi=INV(V)
1790  MAT Wd=Vi*Wb
1800  MAT Wb=Wd
1810  MAT Ui=INV(U)
1820  MAT Wd=Wa
1830  MAT Wa=Wd*Ui
1840  MAT Wc=Wc*(2)
1850  CALL Solve(Wa(*),Wb(*),Wc(*),Wd(*),We(*))
1860  MAT A=Wc*Vi
1870  MAT Wd=TRN(Wc)
1880  MAT B=Wd*Ui
1890  Ap(P,1,1)=A(1,1)
1900  Ap(P,1,2)=A(1,2)
1910  Ap(P,2,1)=A(2,1)
1920  Ap(P,2,2)=A(2,2)
1930  Bp(P,1,1)=B(1,1)
1940  Bp(P,1,2)=B(1,2)
1950  Bp(P,2,1)=B(2,1)
1960  Bp(P,2,2)=B(2,2)
1970  MAT We=A*Wd
1980  MAT U=U-We
1990  MAT We=B*Wc
2000  MAT V=V-We
```

```
2010  Aic(P)=LOG(DET(U))+Fac*P
2020  IF Aic(P)>=Aicmin THEN 2060
2030  Aicmin=Aic(P)
2040  Pbest=P
2050  MAT Ubest=U
2060  IF P=Ia THEN 2220
2070  L=P+1
2080  FOR K=N TO L STEP -1
2090  A1=Y(1,K)
2100  A2=Y(2,K)
2110  B1=Z(1,K-1)
2120  B2=Z(2,K-1)
2130  Z(1,K)=B1-B(1,1)*A1-B(1,2)*A2
2140  Z(2,K)=B2-B(2,1)*A1-B(2,2)*A2
2150  Y(1,K)=A1-A(1,1)*B1-A(1,2)*B2
2160  Y(2,K)=A2-A(2,1)*B1-A(2,2)*B2
2170  NEXT K
2180  CALL Auto(P+2,N,Y(*),Wa(*))
2190  CALL Auto(P+1,N-1,Z(*),Wb(*))
2200  CALL Cross(P+2,N,Y(*),Z(*),Wc(*))
2210  NEXT P
2220  A1=.5*(Ubest(1,2)+Ubest(2,1))
2230  Ubest(1,2)=Ubest(2,1)=A1
2240  SUBEND
2250  !

2260  SUB Pfc(Pmax,Pbest,R(*),Ap(*),Bp(*),Wa(*),Wb(*),Wc(*),Wd(*),Rn(*),R11,R22,
R12)
2270  Rn(1,1,1)=Ap(1,1,1)*R(1,1)+Ap(1,1,2)*R(2,1)
2280  Rn(1,1,2)=Ap(1,1,1)*R(1,2)+Ap(1,1,2)*R(2,2)
2290  Rn(1,2,1)=Ap(1,2,1)*R(1,1)+Ap(1,2,2)*R(2,1)
2300  Rn(1,2,2)=Ap(1,2,1)*R(1,2)+Ap(1,2,2)*R(2,2)
2310  FOR P=2 TO Pbest
2320  Wc(1,1)=Ap(P,1,1)*R(1,1)+Ap(P,1,2)*R(2,1)
2330  Wc(1,2)=Ap(P,1,1)*R(1,2)+Ap(P,1,2)*R(2,2)
2340  Wc(2,1)=Ap(P,2,1)*R(1,1)+Ap(P,2,2)*R(2,1)
2350  Wc(2,2)=Ap(P,2,1)*R(1,2)+Ap(P,2,2)*R(2,2)
2360  FOR L=1 TO P-1
2370  Ib=P-L
2380  Wa(1,1)=Ap(P,1,1)*Bp(Ib,1,1)+Ap(P,1,2)*Bp(Ib,2,1)
2390  Wa(1,2)=Ap(P,1,1)*Bp(Ib,1,2)+Ap(P,1,2)*Bp(Ib,2,2)
2400  Wa(2,1)=Ap(P,2,1)*Bp(Ib,1,1)+Ap(P,2,2)*Bp(Ib,2,1)
2410  Wa(2,2)=Ap(P,2,1)*Bp(Ib,1,2)+Ap(P,2,2)*Bp(Ib,2,2)
2420  Wa(1,1)=Ap(L,1,1)-Wa(1,1)
2430  Wa(1,2)=Ap(L,1,2)-Wa(1,2)
2440  Wa(2,1)=Ap(L,2,1)-Wa(2,1)
2450  Wa(2,2)=Ap(L,2,2)-Wa(2,2)
```

```
2460  Wb(1,1)=Bp(P,1,1)*Ap(L,1,1)+Bp(P,1,2)*Ap(L,2,1)
2470  Wb(1,2)=Bp(P,1,1)*Ap(L,1,2)+Bp(P,1,2)*Ap(L,2,2)
2480  Wb(2,1)=Bp(P,2,1)*Ap(L,1,1)+Bp(P,2,2)*Ap(L,2,1)
2490  Wb(2,2)=Bp(P,2,1)*Ap(L,1,2)+Bp(P,2,2)*Ap(L,2,2)
2500  Bp(Ib,1,1)=Bp(Ib,1,1)-Wb(1,1)
2510  Bp(Ib,1,2)=Bp(Ib,1,2)-Wb(1,2)
2520  Bp(Ib,2,1)=Bp(Ib,2,1)-Wb(2,1)
2530  Bp(Ib,2,2)=Bp(Ib,2,2)-Wb(2,2)
2540  Ap(L,1,1)=Wa(1,1)
2550  Ap(L,1,2)=Wa(1,2)
2560  Ap(L,2,1)=Wa(2,1)
2570  Ap(L,2,2)=Wa(2,2)
2580  Wd(1,1)=Wa(1,1)*Rn(Ib,1,1)+Wa(1,2)*Rn(Ib,2,1)
2590  Wd(1,2)=Wa(1,1)*Rn(Ib,1,2)+Wa(1,2)*Rn(Ib,2,2)
2600  Wd(2,1)=Wa(2,1)*Rn(Ib,1,1)+Wa(2,2)*Rn(Ib,2,1)
2610  Wd(2,2)=Wa(2,1)*Rn(Ib,1,2)+Wa(2,2)*Rn(Ib,2,2)
2620  MAT Wc=Wc+Wd
2630  NEXT L
2640  Rn(P,1,1)=Wc(1,1)
2650  Rn(P,1,2)=Wc(1,2)
2660  Rn(P,2,1)=Wc(2,1)
2670  Rn(P,2,2)=Wc(2,2)
2680  NEXT P
2690  FOR P=Pbest+1 TO Pmax
2700  MAT Wa=ZER
2710  FOR L=1 TO Pbest
2720  Ib=P-L
2730  Wb(1,1)=Ap(L,1,1)*Rn(Ib,1,1)+Ap(L,1,2)*Rn(Ib,2,1)
2740  Wb(1,2)=Ap(L,1,1)*Rn(Ib,1,2)+Ap(L,1,2)*Rn(Ib,2,2)
2750  Wb(2,1)=Ap(L,2,1)*Rn(Ib,1,1)+Ap(L,2,2)*Rn(Ib,2,1)
2760  Wb(2,2)=Ap(L,2,1)*Rn(Ib,1,2)+Ap(L,2,2)*Rn(Ib,2,2)
2770  MAT Wa=Wa+Wb
2780  NEXT L
2790  Rn(P,1,1)=Wa(1,1)
2800  Rn(P,1,2)=Wa(1,2)
2810  Rn(P,2,1)=Wa(2,1)
2820  Rn(P,2,2)=Wa(2,2)
2830  NEXT P
2840  R11=R(1,1)
2850  R22=R(2,2)
2860  R12=R(1,2)
2870  T=SQR(R11*R22)
2880  R(1,1)=R(2,2)=1
2890  R(1,2)=R(2,1)=R12/T
2900  FOR P=1 TO Pmax
2910  Rn(P,1,1)=Rn(P,1,1)/R11
2920  Rn(P,1,2)=Rn(P,1,2)/T
2930  Rn(P,2,1)=Rn(P,2,1)/T
2940  Rn(P,2,2)=Rn(P,2,2)/R22
2950  NEXT P
2960  SUBEND
2970  !
```

```
2980  SUB Pefft(Pbest,Nfft,Ap(*),X11(*),Y11(*),X12(*),X12(*),X21(*),Y21(*),X22(*
),Y22(*))
2990  X11(1)=X22(1)=1
3000  FOR L=1 TO Pbest
3010  X11(L+1)=-Ap(L,1,1)
3020  X12(L+1)=-Ap(L,1,2)
3030  X21(L+1)=-Ap(L,2,1)
3040  X22(L+1)=-Ap(L,2,2)
3050  NEXT L
3060  CALL Fft10(Nfft,X11(*),Y11(*))
3070  CALL Fft10(Nfft,X12(*),Y12(*))
3080  CALL Fft10(Nfft,X21(*),Y21(*))
3090  CALL Fft10(Nfft,X22(*),Y22(*))
3100  SUBEND
3110  !

3120  SUB Sdm(Nfft,Ubest(*),Wa(*),Wb(*),Wc(*),Wd(*),We(*),X11(*),Y11(*),X12(*),Y
12(*),X21(*),Y21(*),X22(*),Y22(*),S11,S22,S12)
3130  T=2*Nfft
3140  S11=S22=S12=0
3150  J=Nfft/2+1
3160  FOR L=1 TO J
3170  Wa(1,1)=X22(L)
3180  Wa(1,2)=-X12(L)
3190  Wa(2,1)=-X21(L)
3200  Wa(2,2)=X11(L)
3210  Wb(1,1)=Y22(L)
3220  Wb(1,2)=-Y12(L)
3230  Wb(2,1)=-Y21(L)
3240  Wb(2,2)=Y11(L)
3250  Ta=DET(Wa)-DET(Wb)
3260  Tb=Wa(1,1)*Wb(2,2)+Wa(2,2)*Wb(1,1)-Wa(1,2)*Wb(2,1)-Wa(2,1)*Wb(1,2)
3270  Ta=T/(Ta*Ta+Tb*Tb)
3280  MAT Wc=TRN(Wa)
3290  MAT Wd=Ubest*Wc
3300  MAT Wc=Wb*Wd
3310  Tb=Wc(1,2)-Wc(2,1)
3320  MAT Wc=Wa*Wd
3330  MAT Wd=TRN(Wb)
3340  MAT We=Ubest*Wd
3350  MAT Wd=Wb*We
3360  MAT Wc=Wc+Wd
3370  Y11(L)=(Wc(1,2)^2+Tb*Tb)/(Wc(1,1)*Wc(2,2))
3380  Y22(L)=FNArg(Wc(1,2),Tb)
3390  X11(L)=Ta*Wc(1,1)
3400  X22(L)=Ta*Wc(2,2)
3410  X12(L)=Ta*Wc(1,2)
3420  Y12(L)=Ta*Tb
3430  S11=S11+X11(L)
3440  S22=S22+X22(L)
3450  S12=S12+X12(L)
3460  NEXT L
3470  S11=S11-.5*(X11(1)+X11(J))
3480  S22=S22-.5*(X22(1)+X22(J))
3490  S12=S12-.5*(X12(1)+X12(J))
3500  SUBEND
3510  !
```

```
3520  SUB Acm(Nfft,X11(*),X12(*),Y12(*),X21(*), Y21(*),Y22(*),X11m1,X22m1,X11m0,X
22m0)
3530  N2=Nfft/2
3540  N21=N2+1
3550  N22=N2+2
3560  FOR L=1 TO N2
3570  X21(L)=.5*X11(L)
3580  Y21(L)=.5*X22(L)
3590  X21(N2+L)=.5*X11(N22-L)
3600  Y21(N2+L)=.5*X22(N22-L)
3610  NEXT L
3620  CALL Fft10(Nfft,X21(*),Y21(*))
3630  Ta=1/X21(1)
3640  Tb=1/Y21(1)
3650  T=SQR(Ta*Tb)
3660  X11(N22)=X22(N22)=1
3670  FOR L=2 TO N2-1
3680  X11(N21+L)=X21(L)*Ta
3690  X22(N21+L)=Y21(L)*Tb
3700  NEXT L
3710  X11m1=X21(N2)*Ta
3720  X22m1=Y21(N2)*Tb
3730  X11m0=X21(N21)*Ta
3740  X22m0=Y21(N21)*Tb
3750  X21(1)=.5*X12(1)*T
3760  Y21(1)=-.5*Y12(1)*T
3770  FOR L=2 TO N2
3780  X21(L)=X12(L)*T
3790  Y21(L)=-Y12(L)*T
3800  X21(N2+L)=Y21(N2+L)=0
3810  NEXT L
3820  X21(N21)=.5*X12(N21)*T
3830  Y21(N21)=-.5*Y12(N21)*T
3840  CALL Fft10(Nfft,X21(*),Y21(*))
3850  SUBEND
3860  !
```

```
3870  SUB Cross(N1,N2,A(*),B(*),C(*)
3880  S11=S12=S21=S22=0
3890  FOR K=N1 TO N2
3900  A1=A(1,K)
3910  A2=A(2,K)
3920  B1=B(1,K-1)
3930  B2=B(2,K-1)
3940  S11=S11+A1*B1
3950  S12=S12+A1*B2
3960  S21=S21+A2*B1
3970  S22=S22+A2*B2
3980  NEXT K
3990  C(1,1)=S11
4000  C(1,2)=S12
4010  C(2,1)=S21
4020  C(2,2)=S22
4030  SUBEND
4040  !

4050  SUB Auto(N1,N2,A(*),B(*))
4060  S11=S12=S22=0
4070  FOR K=N1 TO N2
4080  A1=A(1,K)
4090  A2=A(2,K)
4100  S11=S11+A1*A1
4110  S12=S12+A1*A2
4120  S22=S22+A2*A2
4130  NEXT K
4140  B(1,1)=S11
4150  B(1,2)=B(2,1)=S12
4160  B(2,2)=S22
4170  SUBEND
4180  !

4190  SUB Solve(Wa(*),Wb(*),Wc(*),Wd(*),We(*)
4200  Ta=Wa(1,1)*Wa(2,2)+Wb(1,1)*Wb(2,2)
4210  Tb=DET(Wa)-DET(Wb)
4220  MAT Wd=Wc*Wb
4230  We(1,1)=Wa(2,2)
4240  We(1,2)=-Wa(1,2)
4250  We(2,1)=-Wa(2,1)
4260  We(2,2)=Wa(1,1)
4270  MAT Wa=We*Wc
4280  MAT Wd=Wd+Wa
4290  MAT Wb=Wb*(Ta)
4300  Wb(1,1)=Wb(1,1)+Tb
4310  Wb(2,2)=Wb(2,2)+Tb
4320  MAT We=INV(Wb)
4330  MAT Wc=Wd*We
4340  SUBEND
4350  !
```

```
4360  SUB FFT10(N,W(*),V(*))     ' N  = 2 10 = 1024, N=2 INTEGER
4370  DIM C(1:257)
4380  DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998785456205,.998475580573,.998118112900
4390  DATA .997723066644,.997290456678,.996820299291,.996312612182,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
4400  DATA .991709753669,.990902635428,.990058210262,.989175509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
4410  DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
4420  DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
4430  DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.939459223602,.937339011913,.935183509938,.932992798835
4440  DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
4450  DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
4460  DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
4470  DATA .848120344803,.844853565250,.841554977437,.838224705555,.834862874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
4480  DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
4490  DATA .776888465673,.773010453363,.769103337646,.765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
4500  DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
4510  DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
4520  DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763,.615231590581,.610382806276,.605511041404
4530  DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745887,.565731810784,.560661576197,.555570233020
4540  DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
4550  DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
4560  DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
4570  DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717311,.336889853392
4580  DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
4590  DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058106281,.225083911360,.219101240157
4600  DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068664150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
4610  DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.98017140329GE-1
4620  DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
4630  DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
```

```
4640  READ C(*)
4650  M=1024/N
4660  FOR I=0 TO N/4
4670  C(I+1)=C(M*I+1)
4680  NEXT I
4690  N1=N/4
4700  N2=N1+1
4710  N3=N2+1
4720  N4=N1+N3
4730  Log2n=INT(1.4427*LOG(N)+.5)
4740  FOR I1=1 TO Log2n
4750  I2=2^(Log2n-I1)
4760  I3=2*I2
4770  I4=N/I3
4780  FOR I5=1 TO I2
4790  I6=(I5-1)*I4+1
4800  IF I6<=N2 THEN 4840
4810  N6=-C(N4-I6)
4820  N7=-C(I6-N1)
4830  GOTO 4860
4840  N6=C(I6)
4850  N7=-C(N3-I6)
4860  FOR I7=0 TO N-I3 STEP I3
4870  I8=I7+I5
4880  I9=I8+I2
4890  N8=X(I8)-X(I9)
4900  N9=Y(I8)-Y(I9)
4910  X(I8)=X(I8)+X(I9)
4920  Y(I8)=Y(I8)+Y(I9)
4930  X(I9)=N6*N8-N7*N9
4940  Y(I9)=N6*N9+N7*N8
4950  NEXT I7
4960  NEXT I5
4970  NEXT I1
4980  I1=Log2n+1
4990  FOR I2=1 TO 10                    ! 2^10=1024
5000  C(I2)=1
5010  IF I2>Log2n THEN 5030
5020  C(I2)=2^(I1-I2)
5030  NEXT I2
```

```
5040  k=1
5050  FOR I1=1 TO C(10)
5060  FOR I2=I1 TO C(9) STEP C(10)
5070  FOR I3=I2 TO C(8) STEP C(9)
5080  FOR I4=I3 TO C(7) STEP C(8)
5090  FOR I5=I4 TO C(6) STEP C(7)
5100  FOR I6=I5 TO C(5) STEP C(6)
5110  FOR I7=I6 TO C(4) STEP C(5)
5120  FOR I8=I7 TO C(3) STEP C(4)
5130  FOR I9=I8 TO C(2) STEP C(3)
5140  FOR I10=I9 TO C(1) STEP C(2)
5150  J=I10
5160  IF k>J THEN 5230
5170  A=X(k)
5180  X(k)=X(J)
5190  X(J)=A
5200  A=Y(k)
5210  Y(k)=Y(J)
5220  Y(J)=A
5230  k=k+1
5240  NEXT I10
5250  NEXT I9
5260  NEXT I8
5270  NEXT I7
5280  NEXT I6
5290  NEXT I5
5300  NEXT I4
5310  NEXT I3
5320  NEXT I2
5330  NEXT I1
5340  SUBEND
5350  !

5360  DEF FNArg(X,Y)              ! PRINCIPAL ARG(-2
5370  IF X=0 THEN A=.5*PI*SGN(Y)
5380  IF X<>0 THEN A=ATN(Y/X)
5390  IF X<0 THEN A=A+PI*(1-2*(Y<0))
5400  RETURN A
5410  FNEND
5420  !

5430  SUB Data(N,X(*))
5440  OPTION BASE 1
5450  REDIM X(2,N)
5460  DATA 1,2,6,3,1,1,2,1,4,5,3,2,1,5,6,1,2,4,3,3
5470  DATA 2,1,0,1,5,3,0,1,2,6,2,2,4,2,3,5,6,9,8,2
5480  READ X(*)
5490  SUBEND
```

```
NUMBER OF DATA POINTS IN EACH PROCESS N = 20
MAXIMUM ORDER OF PREDICTIVE FILTER Pmax = 6
SIZE OF FFT Nfft = 256

PROCESS NUMBER 1
 1  2  6  3  1  1  2  1  4  5  3  2  1  5  6  1  2  4  8  9

PROCESS NUMBER 2
 2  1  0  1  5  3  0  1  2  6  2  2  4  2  3  5  6  9  8  2


MEANS OF INPUT DATA (Ave):
 3.35
 3.2


COVARIANCE MATRIX OF INPUT DATA (R):
 5.7275                .83

 .83                   6.16


AKAIKE INFORMATION CRITERION:
  P        Aic(P)
  0     .354363690E+01
  1     .324147848E+01
  2     .313209545E+01
  3     .342425965E+01
  4     .235777558E+01
  5     .248589498E+01
  6     .266117185E+01


Pbest = 4


Ubest:
 .54593344383          .226987553923

 .226987553923         4.00240808163


FORWARD PARTIAL CORRELATION COEFFICIENTS:
  P        Ap(1,1)             Ap(2,1)             Ap(1,2)             Ap(2,2)
  1     .326847861E+00     -.296055029E-01     .433508958E+00      .492296101E+00
  2    -.497079412E+00      .851055338E-01     .177974150E+00     -.275017512E+00
  3    -.172083146E+00      .237335358E+00     .586121047E-01      .154803723E+00
  4    -.143631688E+00     -.110686655E+00     .734743629E+00     -.124715363E+00
  5    -.516644665E-01     -.440540782E+00     .193329089E+00      .883560194E+00
  6     .901306062E-01     -.736616304E-01    -.256580155E+00      .734752737E+00


BACKWARD PARTIAL CORRELATION COEFFICIENTS:
  P        Bp(1,1)             Bp(2,1)             Bp(1,2)             Bp(2,2)
  1     .391694481E+00      .451665794E+00    -.139717614E-01      .427449481E+00
  2    -.657848877E+00      .178422853E+00     .271402210E-01     -.192143512E+00
  3    -.218767381E+00      .685350177E-01     .177465355E+00      .129759954E+00
  4    -.128639908E+00      .100617477E+01    -.865594759E-01     -.944141856E-01
  5    -.257137623E-01      .194124206E+00    -.219204308E+00      .117920510E+00
  6     .182731463E+00     -.284468163E+00     .626220731E-01      .161491679E+00
```

FORWARD PREDICTIVE FILTER COEFFICIENTS FOR Pbest:

P	Ap(1,1)	Ap(2,1)	Ap(1,2)	Ap(2,2)
1	.235726249E+00	.174118149E+00	.246570575E+00	.670170572E+00
2	-.600483360E+00	-.164936772E+00	.412699679E+00	-.442413930E+00
3	-.404360932E+00	.342089409E+00	-.235747642E+00	.251445573E+00
4	-.143631688E+00	-.110686655E+00	.734743429E+00	-.124715363E+00

NORMALIZED CORRELATION MATRICES (Rn):

DELAY	AUTO11	CROSS21	CROSS12	AUTO22
0	.100000000E+01	.139734996E+00	.139734996E+00	.100000000E+01
1	.389669762E+00	.402437203E-01	.495250946E+00	.488307047E+00
2	-.284751305E+00	.912480910E-01	.511090660E+00	.640872368E-01
3	-.372945922E+00	.195208145E+00	.117107113E+00	.663036688E-01
4	-.223941701E-01	.876238662E-01	.190792716E+00	.100976458E+00
5	.396697360E+00	-.896791098E-01	.112701803E+00	.122691975E+00
6	.330388400E+00	-.836262522E-01	-.951969127E-01	.188886781E-01

SPECTRAL DENSITY MATRIX AND COHERENCE, FROM ZERO FREQUENCY (BIN 1):

BIN	AUTO11	AUTO22	RE(CROSS12)	IM(CROSS12)	MAG SQ COH	ARGUMENT
1	.563234E-01	.130480E+00	.848704E-01	.000000E+01	.980117E+00	.000000E+01
2	.562462E-01	.130295E+00	.847113E-01	-.254607E-02	.980064E+00	-.300468E-01
3	.560152E-01	.129740E+00	.842359E-01	-.506959E-02	.979904E+00	-.601108E-01
4	.556327E-01	.128822E+00	.834493E-01	-.754840E-02	.979634E+00	-.902094E-01
5	.551025E-01	.127549E+00	.823603E-01	-.996105E-02	.979250E+00	-.120360E+00
6	.544301E-01	.125934E+00	.809811E-01	-.122873E-01	.978746E+00	-.150581E+00
7	.536226E-01	.123994E+00	.793277E-01	-.145083E-01	.978113E+00	-.180892E+00
8	.526890E-01	.121751E+00	.774195E-01	-.166075E-01	.977342E+00	-.211311E+00
9	.516400E-01	.119230E+00	.752793E-01	-.185704E-01	.976419E+00	-.241858E+00
10	.504882E-01	.116460E+00	.729330E-01	-.203857E-01	.975331E+00	-.272557E+00
11	.492476E-01	.113474E+00	.704088E-01	-.220448E-01	.974059E+00	-.303429E+00
12	.479335E-01	.110309E+00	.677369E-01	-.235426E-01	.972584E+00	-.334499E+00
13	.465623E-01	.107002E+00	.649485E-01	-.248774E-01	.970883E+00	-.365795E+00
14	.451507E-01	.103593E+00	.620751E-01	-.260508E-01	.968928E+00	-.397145E+00
15	.437158E-01	.100121E+00	.591473E-01	-.270676E-01	.966688E+00	-.429181E+00
***						
116	.495654E-02	.527756E-02	.180088E-02	.366875E-02	.638529E+00	.111448E+01
117	.464126E-02	.492382E-02	.190282E-02	.323458E-02	.616259E+00	.103904E+01
118	.436922E-02	.461992E-02	.198172E-02	.284179E-02	.594634E+00	.961851E+00
119	.413522E-02	.435955E-02	.204244E-02	.248491E-02	.573915E+00	.882822E+00
120	.393482E-02	.413733E-02	.208689E-02	.215898E-02	.554351E+00	.801897E+00
121	.376422E-02	.394877E-02	.212416E-02	.185948E-02	.536176E+00	.719053E+00
122	.362025E-02	.379007E-02	.215073E-02	.158236E-02	.519605E+00	.634308E+00
123	.350027E-02	.365814E-02	.217053E-02	.132398E-02	.504835E+00	.547727E+00
124	.340211E-02	.355042E-02	.218509E-02	.108104E-02	.492036E+00	.459427E+00
125	.332403E-02	.346487E-02	.219558E-02	.850527E-03	.481358E+00	.369582E+00
126	.326465E-02	.339991E-02	.220289E-02	.629679E-03	.472923E+00	.278418E+00
127	.322295E-02	.335433E-02	.220767E-02	.415919E-03	.466828E+00	.186214E+00
128	.319821E-02	.332731E-02	.221037E-02	.206809E-03	.463144E+00	.932912E-01
129	.319001E-02	.331836E-02	.221125E-02	.000000E+01	.461911E+00	.000000E+01

TRAPEZOIDAL SUMS OF SPECTRA:

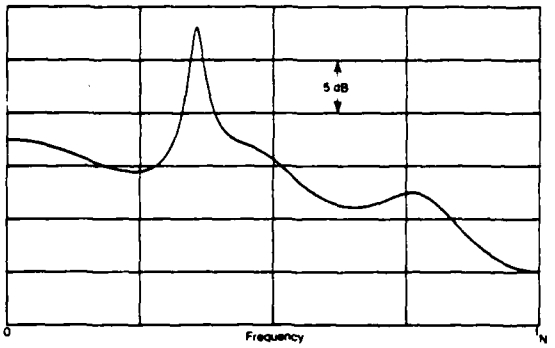
5.72678260464	6.15993376582	.830232954125

COVARIANCES OF INPUT DATA:

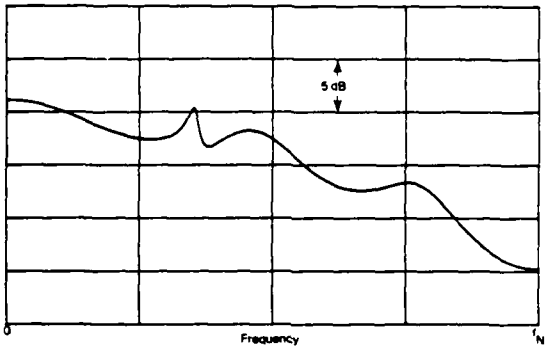
5.7275	6.16	.83

BLIASED NORMALIZED CORRELATION MATRICES:

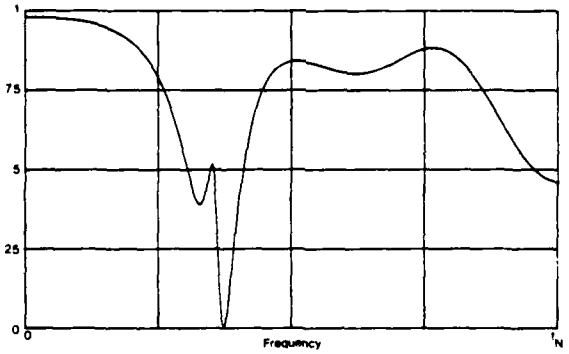
DELAY	AUTO11	CROSS21	CROSS12	AUTO22
0	.100000000E+01	.139783721E+00	.139783721E+00	.100000000E+01
1	.389662823E+00	.402483863E-01	.495316301E+00	.488307810E+00
2	-.284711075E+00	.912157098E-01	.511114181E+00	.640949626E-01
3	-.372868877E+00	.195185513E+00	.117074400E+00	.663147971E-01
4	-.223628911E-01	.876380084E-01	.190781841E+00	.100979179E+00
5	.396652516E+00	-.896408926E-01	.112729049E+00	.122684113E+00
6	.330310631E+00	-.836012969E-01	-.951530121E-01	.188794710E-01
7	-.345026269E-01	.609690818E-02	-.999179326E-01	-.796893320E-02
8	-.308860295E+00	.824851230E-01	.131798148E-01	.197442248E-01
9	-.262499156E+00	.629998356E-01	.1.6921946E+00	-.191113963E-01
10	.795870936E-01	-.238734308E-01	.106767456E+00	-.211019705E-01
11	.310325251E+00	-.669643776E-01	-.976171226E-01	.506845393E-02
12	.192490120E+00	-.429532422E-01	-.148308163E+00	.290497257E-01
13	-.123481195E+00	.241857986E-01	-.432258645E-01	.212691423E-01
14	-.294362011E+00	.638641478E-01	.104657982E+00	-.219543907E-01
15	-.128164922E+00	.302603975E-01	.123725501E+00	-.315105569E-01
***				
116	-.244194112E-02	.913297510E-03	.313869734E-02	-.782794585E-03
117	.270370651E-02	.703065771E-03	.135635926E-02	-.620489604E-03
118	.411244195E-02	-.122243230E-03	-.167993100E-02	.177358396E-03
119	.805212010E-03	-.844466456E-03	-.265889725E-02	.750518242E-03
120	-.280736944E-02	-.822305920E-03	-.764920706E-03	.519056852E-03
121	-.264788177E-02	-.179269851E-04	.172870237E-02	-.239248236E-03
122	.440106016E-03	.889991252E-03	.216969761E-02	-.719997787E-03
123	.238535013E-02	.101284639E-02	.329164628E-03	-.439526451E-03
124	.112686107E-02	.992673477E-04	-.164236260E-02	.286993480E-03
125	-.121802675E-02	-.105565279E-02	-.171669788E-02	.698618350E-03
126	-.151757804E-02	-.124026530E-02	-.491074185E-04	.378592478E-03
127	.306556500E-03	-.866410300E-04	.146460666E-02	-.330374132E-03
128	.148247837E-02	.133650385E-02	.133650385E-02	-.690972582E-03



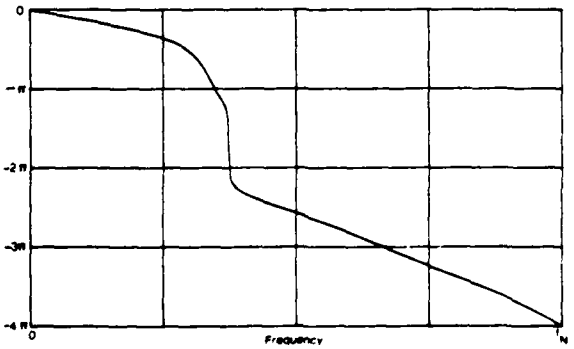
A-1A. Auto Spectrum of First Process



A-1B. Auto Spectrum of Second Process



A-1C. Magnitude-Squared Coherence



A-1D. Argument of Complex Coherence

Figure A-1. Spectral Estimates for N = 20, Pmax = 6, Pbest = 4

Appendix B

General Filter and Spectral Relations

For each integer k, let H_k be a rectangular matrix of (complex) constants, to be called the filter impulse response at delay $k\Delta$, where Δ is the time sampling increment. The number of filter outputs need not equal the number of filter inputs. For multichannel filter excitation W_n at time $n\Delta$, the filter output at time $n\Delta$ is given by the discrete convolution

$$X_n = \sum_k H_k \, W_{n-k} \quad , \tag{B-1}$$

where the summation extends over all nonzero summands.

For a stationary excitation, let the correlation matrix of complex input process $\{W_n\}$ at delay $\ell\Delta$ be

$$\overline{W_n \, W_{n-\ell}^H} = P_\ell \quad \text{(non-diagonal matrix)}, \tag{B-2}$$

where the overbar denotes an *ensemble average*, and the superscript H denotes a conjugate transpose. The z-transform of input correlation sequence $\{P_\ell\}$ is

$$Q(z) \equiv \Delta \sum_\ell z^{-\ell} \, P_\ell \quad , \tag{B-3}$$

and the spectrum of input process $\{W_n\}$ is, for real frequency f,

$$\begin{aligned} Q(f) = Q(\exp(i2\pi f\Delta)) &= \Delta \sum_\ell \exp(-i2\pi f\Delta\ell) \, P_\ell \\ &= \Delta \sum_\ell \exp(-iu\ell) \, P_\ell \quad , \end{aligned} \tag{B-4}$$

where we let

$$u \equiv 2\pi f\Delta \quad . \tag{B-5}$$

The correlation matrix of the filter output process $\{X_n\}$, at delay $\ell\Delta$, is given by using (B-1) and (B-2):

$$R_\ell \equiv \overline{X_n \, X_{n-\ell}^H} = \sum_{k,m} H_k \, \overline{W_{n-k} \, W_{n-\ell-m}^H} \, H_m^H = \sum_{k,m} H_k \, P_{\ell+m-k} \, H_m^H \quad . \tag{B-6}$$

The z-transform of output correlation sequence $\{R_\ell\}$ is, by use of (B-6),

$$G(z) \equiv \Delta \sum_{\ell} z^{-\ell} R_{\ell} = \Delta \sum_{\ell} z^{-\ell} \sum_{k,m} H_k \, P_{\ell+m-k} \, H_m^H$$

$$= \sum_{k} z^{-k} \, H_k \, \Delta \sum_{\ell} z^{-(\ell+m-k)} \, P_{\ell+m-k} \sum_{m} z^{m} \, H_m^H \quad . \qquad \text{(B-7)}$$

Now define the z-transform of filter sequence $\{H_k\}$ by

$$H(z) = \sum_{k} z^{-k} \, H_k \qquad \text{(B-8)}$$

and define the quantity

$$H^H(z) \equiv [H(z)]^H \quad , \quad \text{including conjugation of } z. \qquad \text{(B-9)}$$

Then

$$H^H\left(\frac{1}{z^*}\right) = \left[H\left(\frac{1}{z^*}\right)\right]^H = \left[\sum_{k} z^{*k} \, H_k\right]^H = \sum_{k} z^k \, H_k^H \quad . \qquad \text{(B-10)}$$

We now employ (B-8), (B-3), and (B-10) in (B-7) to obtain

$$G(z) = H(z) \, Q(z) \, H^H\left(\frac{1}{z^*}\right) \quad . \qquad \text{(B-11)}$$

The spectrum of output process $\{X_n\}$ is then, at real frequency f,

$$G(f) = G(\exp(i2\pi f\Delta)) = \Delta \sum_{\ell} \exp(-i2\pi f\Delta\ell) \, R_{\ell}$$

$$= H(f) \, Q(f) \, H^H(f) \quad , \qquad \text{(B-12)}$$

where we employed (B-11), (B-4), and set

$$H(f) = H(\exp(i2\pi f\Delta)) = \sum_{k} \exp(-i2\pi f\Delta k) \, H_k \quad . \qquad \text{(B-13)}$$

We also employed the property that

$$H^H(f) = [H(f)]^H = \sum_{k} \exp(i2\pi f\Delta k) \, H_k^H \quad , \qquad \text{(B-14)}$$

or

$$H^H\left(\frac{1}{\exp(-i2\pi f\Delta)}\right) = H^H(\exp(i2\pi f\Delta)) = H^H(f) \quad . \tag{B-15}$$

Finally, from convolution (B-1), we obtain the z-transform of output data sequence $\{X_n\}$ as

$$X(z) \equiv \sum_n z^{-n} X_n = H(z)\ W(z) \quad , \tag{B-16}$$

where we used (B-8) and defined

$$W(z) = \sum_n z^{-n} W_n \quad . \tag{B-17}$$

The major results thus far are given by (B-1), (B-16), (B-11), and (B-12) for a general filter and excitation.

ARMA Process

For a multichannel, autoregressive, moving-average process, the recursion is given by

$$X_n = \sum_k E_k X_{n-k} + \sum_k F_k W_{n-k} \quad . \tag{B-18}$$

The z-transform of this equation is

$$X(z) = E(z)\ X(z) + F(z)\ W(z) \quad , \tag{B-19}$$

where

$$E(z) \equiv \sum_k z^{-k} E_k \quad , \quad F(z) \equiv \sum_k z^{-k} F_k \quad . \tag{B-20}$$

Then we can solve (B-19) for $X(z)$ as

$$X(z) = [I - E(z)]^{-1} F(z)\ W(z) = H(z)\ W(z) \quad , \tag{B-21}$$

where the transfer function from input to output is

$$H(z) \equiv [I - E(z)]^{-1} F(z) \tag{B-22}$$

in terms of the parameters of recursion (B-18). But now (B-22) is in the framework of the presentation above; namely, the spectrum of output process $\{X_n\}$ is, from (B-12),

$$G(f) = H(f)\ Q(f)\ H^H(f) \quad , \tag{B-23}$$

where

$$H(f) = [I - E(f)]^{-1} F(f) \quad ,$$

$$E(f) = E(\exp(i2\pi f\Delta)) = \sum_k \exp(-i2\pi f\Delta k)\, E_k \quad ,$$

$$F(f) = F(\exp(i2\pi f\Delta)) = \sum_k \exp(-i2\pi f\Delta k)\, F_k \quad . \tag{B-24}$$

Example

As an example of (B-18), consider the multichannel first-order recursion

$$X_n = E_1\, X_{n-1} + W_n \tag{B-25}$$

with the input spectrum for $\{W_n\}$,

$$Q(f) = \Delta\, I \quad \text{for all} \quad f \quad . \tag{B-26}$$

This is a white process, uncorrelated from channel to channel. Then

$$E(z) = z^{-1} E_1 \quad , \quad F(z) = I \quad ,$$

$$H(z) = (I - z^{-1} E_1)^{-1} \quad , \tag{B-27}$$

$$G(f) = \Delta\, H(f)\, H^H(f) \quad .$$

Specialization to Two-Channel Process

We further specialize example (B-25) to the two-input, two-output channel process characterized by coefficient matrix

$$E_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{(complex coefficients).} \tag{B-28}$$

Then (B-27) and (B-22) yield transfer function

$$H(z) = \begin{bmatrix} 1 - z^{-1} a & -z^{-1} b \\ -z^{-1} c & 1 - z^{-1} d \end{bmatrix}^{-1}$$

$$= \frac{1}{1 - z^{-1}(a + d) + z^{-2}(ad - bc)} \begin{bmatrix} 1 - z^{-1} d & z^{-1} b \\ z^{-1} c & 1 - z^{-1} a \end{bmatrix} . \quad \text{(B-29)}$$

There follows

$$H^H(1/z^*) = [H(1/z^*)]^H$$

$$= \frac{1}{1 - z(a + d)^* + z^2(ad - bc)^*} \begin{bmatrix} 1 - zd^* & zc^* \\ zb^* & 1 - za^* \end{bmatrix} \quad \text{(B-30)}$$

and

$$G(z) = \Delta\, H(z)\, H^H(1/z^*) \equiv \begin{bmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{bmatrix}$$

$$= \frac{1}{D} \begin{bmatrix} 1 + |b|^2 + |d|^2 - z^{-1} d - zd^* & -a^* b - c^* d + z^{-1} b + zc^* \\ -ab^* - cd^* + z^{-1} c + zb^* & 1 + |a|^2 + |c|^2 - z^{-1} a - za^* \end{bmatrix} , \quad \text{(B-31)}$$

where

$$D = \left[1 - z^{-1}(a + d) + z^{-2}(ad - bc)\right] \left[1 - z(a + d)^* + z^2(ad - bc)^*\right] . \quad \text{(B-32)}$$

Inspection of denominator D in (B-32) reveals that $G(z)$ has poles (i.e., all the elements of matrix $G(z)$ have poles) at

$$z_\infty = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} \tag{B-33}$$

and at

$$\frac{1}{z_\infty^*} = \frac{1}{|z_\infty|} \exp(i \arg(z_\infty)) \ . \tag{B-34}$$

That is, even though recursion (B-25) is limited to first-order regressive and white independent excitations (see (B-26)), $G(z)$ has *four* poles in the finite z-plane. Generally, if $H(z)$ has a pole at z_∞, then $H(1/z^*)$ has a pole at $1/z_\infty^*$; so if

$$z_\infty = r \exp(i\theta), \quad \text{then} \quad 1/z_\infty^* = \exp(i\theta)/r \tag{B-35}$$

has the same argument.

In addition, element $g_{11}(z)$ in (B-31) has a zero at ∞ and three zeros in the finite z-plane, at

$$z_0 = 0 \quad \text{and} \quad z_0 = \frac{1 + |b|^2 + |d|^2 \pm \sqrt{\left(1 + |b|^2 + |d|^2\right)^2 - 4|d|^2}}{2d^*} \tag{B-36}$$

The product of the latter two zero locations is $d/d^* = \exp(i2\arg(d))$. Thus, the auto spectrum of process 1 has three zeros and four poles, even though the multichannel recursion, (B-25), is first-order regressive. Similar behavior is true of the auto spectrum of the second process, as well as the cross spectrum between the two processes.

The magnitude-squared coherence for this example is, from (B-31) and (B-5),

$$\frac{g_{12}(\exp(iu))\ g_{21}(\exp(iu))}{g_{11}(\exp(iu))\ g_{22}(\exp(iu))}$$

$$= \frac{\left|- a^*b - c^*d + b \exp(-iu) + c^* \exp(iu)\right|^2}{\left[|1 - d \exp(-iu)|^2 + |b|^2\right]\left[|1 - a \exp(- iu)|^2 + |c|^2\right]} . \tag{B-37}$$

This has four zeros and four poles in the finite z-plane.

Numerical Example

We now specialize (B-28) to example (6) in the main text:

$$E_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} .85 & -.75 \\ .65 & .55 \end{bmatrix} . \qquad \text{(B-38)}$$

The four poles of the spectrum $G(z)$ are located at

$$z_\infty = .7 \pm i\ .6819\ , \qquad 1/z_\infty^* = .7330 \pm i\ .7140\ , \qquad \text{(B-39)}$$

and the zeros of $g_{11}(z)$ are at

$$z_0 = 0\ ,\ \ 3.0646,\ \ .3263,\ \infty\ . \qquad \text{(B-40)}$$

The zeros of $g_{12}(z)$ are located instead at

$$z_0 = 0\ ,\ .8802,\ \ -1.3109,\ \ \infty\ . \qquad \text{(B-41)}$$

The magnitude-squared coherence simplifies to

$$\frac{1.0634 - .056\ \cos(u) - .975\ \cos(2u)}{[1.865 - 1.1\ \cos(u)]\ [2.145 - 1.7\ \cos(u)]}\ . \qquad \text{(B-42)}$$

This example was used frequently in the main body of this report. The peak value is .9990128 at u = .772564, or $2f\Delta$ = .245915.

References

1. J.P. Burg, "A New Analysis Technique for Time Series Data," *NATO Advanced Study Institute on Signal Processing*, Enschede, Netherlands, vol. 1, August 1968.

2. R.T. Lacoss, "Data-Adaptive Spectral Analysis Methods," *Geophysics*, vol. 36, no. 4, August 1971, pp. 661-675.

3. J. Makhoul, "Linear Prediction: A Tutorial Review," *Proceedings of the IEEE*, vol. 63, no. 4, April 1975, pp. 561-580.

4. A.H. Nuttall, "Spectral Analysis of a Univariate Process with Bad Data Points, via Maximum Entropy and Linear Predictive Techniques," NUSC Technical Report 5303, 26 March 1976.

5. S.L. Marple, "A New Autoregressive Spectrum Analysis Algorithm," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. ASSP-28, no. 4, August 1980, pp. 441-454.

6. A.H. Nuttall, "Application of Linear Predictive Spectral Analysis to Multiple Tones in Noise," NUSC Technical Memorandum 791218, 12 December 1979.

7. A.H. Nuttall, *FORTRAN Program for Multivariate Linear Predictive Spectral Analysis, Employing Forward and Backward Averaging*, NUSC Technical Document 5419, 19 May 1976.

8. A.H. Nuttall, *Multivariate Linear Predictive Spectral Analysis, Employing Weighted Forward and Backward Averaging: A Generalization of Burg's Algorithm*, NUSC Technical Report 5501, 13 October 1976.

9. A.H. Nuttall, *Positive Definite Spectral Estimate and Stable Correlation Recursion for Multivariate Linear Predictive Spectral Analysis*, NUSC Technical Report 5729, 14 November 1977.

10. S.L. Marple, personal communication.

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